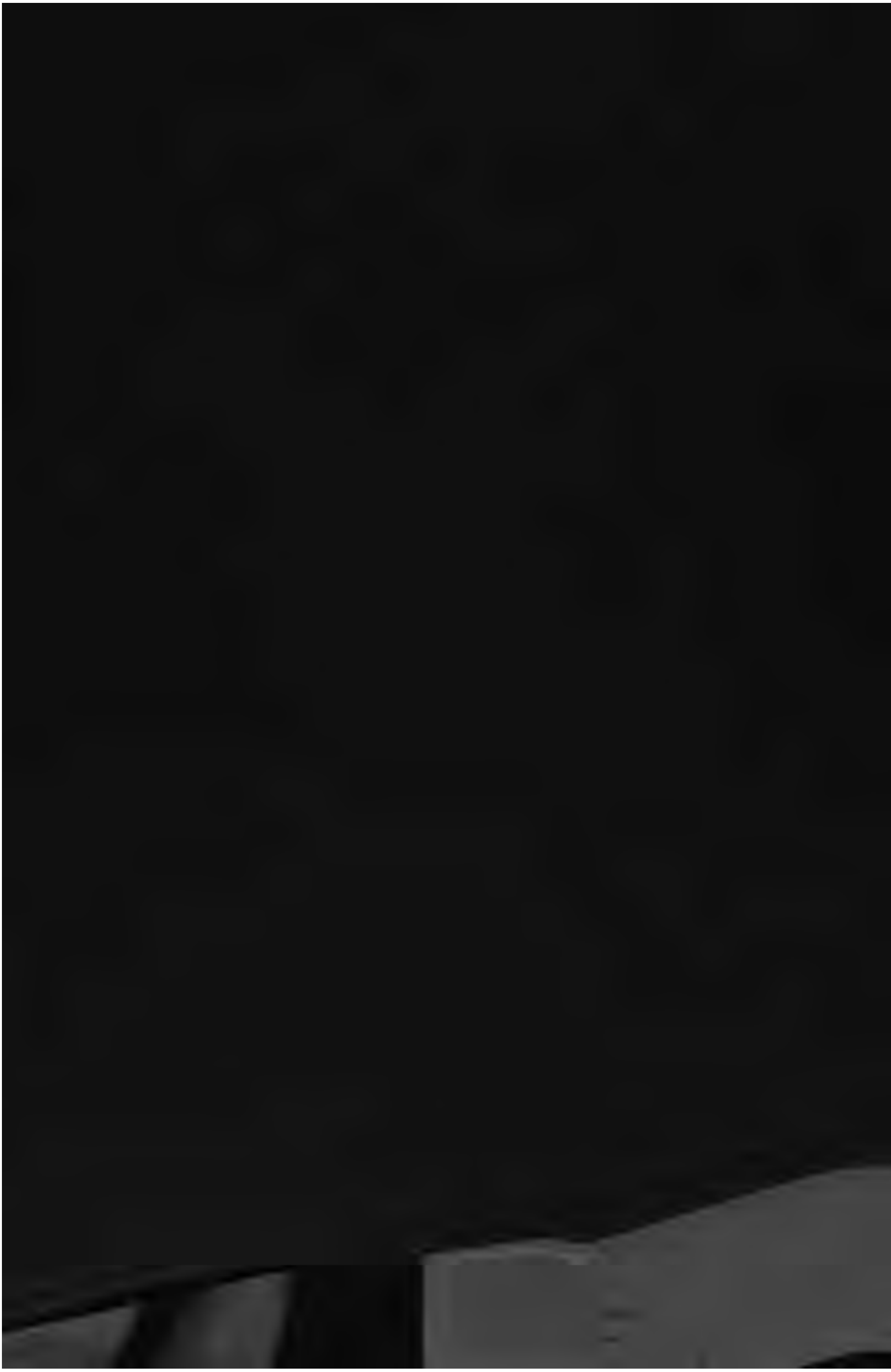

THEORY OF ELECTRICITY AND MAGNETISM.

THEORY OF ELECTRICITY AND MAGNETISM

BY

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WITH A PREFACE BY

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PREFACE BY LUDWIG BOLTZMANN.

INVITED by my German students I wrote a book* giving an extract of some of my lectures on Electricity and Magnetism, held at the University of Munich during the winter and summer terms of 1892-93. I omitted in this book for the sake of brevity most of the illustrations and examples and made the deductions as short as possible, so that I did not give a complete treatise but rather only marked the points, where my ideas of treating the subject differed from those of other scientists.

Dr. Curry thought that this book translated into English might be useful to English and American students, but he has found it better to offer them a more elaborate and complete treatise; in this he has retained more or less the method of treatment and the order of my verbal lectures, has supplied the necessary examples and more elaborate deductions for illustrating the subject-matter, and lastly has inserted not only those of my verbal lectures, which found no place in my book, as the theory of the electric (Hertzian) oscillations, Maxwell's equations for moving bodies, etc., but several

* *Vorlesungen über Maxwell's Theorie der Electricität und des Lichtes.* Johann Barth, Leipzig, Part I. 1891, Part II. 1893.

new articles (§§ 19, 21, etc.), so that the present treatise differs in many respects from both my own book and my verbal lectures.

This treatise, the manuscript of which I have revised, gives a very clear and concise exposition not only of all my lectures on Electricity and Magnetism but especially of the conceptions and principles embodied in the foundation of the electro-magnetic theory of light.

LUDWIG BOLTZMANN.

VIENNA, 1897.

GENERAL PREFACE.

IT has been my desire to embody in the present treatise not only the standpoint, from which the theory of electricity and magnetism is studied on the continent, but also the general methods of treatment in vogue; hereby I naturally include an interpretation of the views set forth in Maxwell's treatise. In starting from given fundamental expressions, in forming certain conceptions and making various assumptions concerning the ether, in formulating thus the theory of electricity and magnetism and in seeking then to derive therefrom and explain thereby all electric and magnetic phenomena, I may be following too closely in the footsteps of the ancient philosophers, who attempted to treat all problems each according to his own system of philosophy; I admit however that this has been my constant aim—for this reason it would perhaps have been more appropriate to have entitled this treatise the *philosophy* of electricity and magnetism. In endeavouring to attain this end I have found it necessary to maintain a sharp distinction between the ether—its real nature and given properties—that is, between Maxwell's equations of action, which define the state of the ether, and the several mechanical or dynamical analogies constantly employed to illustrate

the manifold phenomena and properties expressed by certain particular integrals of these equations. These dynamical illustrations united under the name of our concrete representation have often nothing in common with our conceptions of the ether itself; the very conception or definition of so-called real electricity belongs indeed to the former class (cf. p. 47). On the other hand, the fact that our concrete representation contains so many different features not only justifies our avoiding any attempt to form a more definite conception of the ether, but, conversely, it offers an explanation for the difficulty encountered in grasping the ether-agent itself. The analogy between the different states of the ether and the vibrating elastic band of § 11 does not strictly belong to our concrete representation; it has merely been introduced for the purpose of offering a means of classifying given states of the ether or classes of particular integrals corresponding to these states.

The acceptance of Ampère's assumption of molecular currents as an explanation of magnetic phenomena and of the new definition for α, β, γ, thereby necessary, has been required by the fundamental expressions and conceptions of the first chapter.

Lastly, I have inserted two rather long chapters on von Helmholtz's theory of electricity and magnetism in order that the student might become acquainted with a more general theory than Maxwell's, to which he could have recourse, in case it ever became necessary to abandon the latter on account of its failure to explain phenomena that might be discovered in the future; the universal interest taken in all ether-oscillations since the discovery of the Röntgen rays instigated here a short

examination of the longitudinal waves (gravitation) peculiar to von Helmholtz's ether and of ether-oscillations in general. In concluding this treatise with a brief and somewhat unfinished introduction into the theory of electro- and magneto-striction, I shall feel that I am accomplishing my end, provided it only awakens a desire in the student to pursue this extensive and interesting subject.

This treatise has been written rather elaborately and in detail, in the form of a text-book, which purpose I hope it may serve. It is intended however as only a purely theoretical work, and as such all data, description of experiments, etc., have been omitted.

I am greatly indebted to Professor Boltzmann not only for the kind interest he has shown in the progress of this work, but for his inestimable assistance during its preparation. My thanks are also due to J. C. Beattie, Sc.D., F.R.S.(Ed.), for a careful revision of the proofs and for many valuable suggestions.

C. E. CURRY.

MUNICH,
March, 1897.

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CHAPTER I.

SECTION I. INTRODUCTION; FUNDAMENTAL EXPRESSIONS AND CONCEPTIONS.

ALL branches of theoretical physics, with the exception of electricity and magnetism, can be regarded at the present state of science as concluded, that is, only immaterial changes occur in them from year to year. This cannot, however, be maintained of the branch in question, for each phenomenon or group of phenomena has always been, and in many cases continues even now to be, studied separately and independently, and a theory presented for its explanation without any reference whatever to the other phenomena or groups. First came Coulomb's law of the action between electrified bodies; then Green's and Thomson's theory on the distribution of electricity in insulated bodies based upon this law—these phenomena are at present included under the name of electrostatics—and an analogous theory for the distribution of magnetism by Poisson. These were followed by the theory of stationary flow from Ohm and Weber, then that of electro-magnetism from Oerstedt and Ampère, and finally, that of induction from Franz Neumann. At that period each group of phenomena formed a concrete whole, and each theory was established entirely independent of the other theories; there was, indeed, no attempt whatever made to connect in any way either

the different groups of phenomena or the theories presented for their explanation. The first such attempt came later from Weber; he conceived two electric fluids, a positive and a negative, which acted on each other according to the inverse square of the distance.

The assumption of Weber's law, or, indeed, of any direct action at a distance, has always been a question of controversy, for daily experience teaches us that we cannot alter the state of motion of a body without either directly touching it or at least placing ourselves in communication with it by means of an intervening medium which is capable of imparting the impulse in question. Consequently, when it was first observed that magnetic poles, and later, electrified bodies, acted upon one another apparently at a distance, the only explanation was that the action was conveyed by an intervening medium, as a thin fluid, supposed to be emitted from these bodies.* For the same reason Newton encountered so many obstacles in establishing his law of gravitation; for although there is little doubt that he really believed that the apparent action at a distance between the heavenly bodies was conveyed by a medium, he was, nevertheless, regarded as ascribing it to a direct action at a distance, simply because he was unable to form any satisfactory conception of such a medium.

It was, perhaps, the similarity between the laws of magnetic and electric phenomena and those of gravitation that led Weber to attribute the former to a direct action at a distance. Navier, Poisson, and others even attempted to explain molecular forces in a similar manner, assuming an unknown function of the distance instead of Newton's, whereas Zöllner went so far as to maintain that Weber's function of action at a distance was the key to all problems of nature. The

* Gilbert, *De magnete*, etc., London, 1600, liber II., capp. iii. and iv.

shortcomings of Weber's theory appeared, however, as time advanced.

It is interesting to observe that the reaction from the theory of action at a distance to the assumption of a medium came from England, where, strictly speaking, the former started. This new period began with Faraday; he fully disbelieved in every unaided action at a distance, and assumed a medium, the so-called ether, which was supposed to pervade not only all ponderable bodies, but entire space—its properties were naturally modified by the presence of the former—and to be the transmitter of all electric and magnetic effects. These entirely new and strange conceptions concerning electricity and magnetism had only partially matured in Faraday's mind; their further development was left to Lord Kelvin and others. Then came Clerk Maxwell, who clothed these conceptions in mathematical form, and thus presented to the world a theory which, however great the productions of his followers, will always justly be called Maxwell's theory of electricity and magnetism. This was the second attempt to present a concrete theory that should explain all phenomena of electricity and magnetism and include within it all the theories required for the explanation of the several groups of phenomena. It is the aim of this book to show that Maxwell's theory, with its recent modifications and developments, suffices to explain all phenomena of electricity and magnetism, and, on the other hand, that all electric and magnetic phenomena follow directly from it.

It was many years before Maxwell's theory received general recognition, in fact, not until the experiments of Hertz had been verified. These experiments turned the current in favour of Maxwell's theory, and since then it has achieved triumph upon triumph; the experiments themselves were, however, its greatest triumph, perhaps, indeed, the greatest ever accorded any theory.

In accepting Maxwell's theory one of the first ques-

tions asked is: How is this ether or medium constituted, or what are its several properties? Instead of attempting to answer this and similar questions, and of examining all possible conceptions of such an ether here at the outset, we shall follow essentially Lord Kelvin's method of treating this subject; we shall start, namely, from Maxwell's fundamental equations, interpret these equations as so-called quasi-rigid equations, ascribe only those properties to the ether that are indispensable, that is, that necessarily follow from Maxwell's equations interpreted as such, and shall leave it to the student to form any conception whatever consistent with these few essential properties.

In accepting Maxwell's theory we are excluding direct action at a distance; the change in the state of any volume-element of the ether during any element of time will thus be determined uniquely by the conditions which prevailed at the beginning of that time in the immediate neighbourhood of that volume-element. On account of being unable to offer an entirely satisfactory mechanical explanation of how this action is conveyed from element to element, we must be contented for the present with conceptions of a rather general and vague character.

We assume that a motion, the nature of which is unknown to us, exists in every volume-element of the ether, and that the displacement produced by this motion can be resolved in the three components F, G, H along the coordinate axes, just as the rectilinear displacement of a particle can be resolved along three axes. We can then represent this displacement geometrically by a straight line, called the vector, of such length and direction that its projections on the three coordinate axes are F, G, H. This vector (F, G, H) is called by Faraday the electrotonic or tonic state, or the tone of the volume-element in question, and F, G, H its components. Let us denote the rate of change of the components of this tonic vector by P, Q, R, putting

$$P=\frac{dF}{dt},\quad Q=\frac{dG}{dt},\quad R=\frac{dH}{dt}.^{*}\ \ldots\ldots\ldots\ldots\ldots(1)$$

We shall denote by $Td\tau$ the kinetic energy of the tonic motion in the volume-element $d\tau$. In isotropic bodies let

$$T=\frac{K}{8\pi}(P^2+Q^2+R^2),\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where for any given body the quantity K is a constant, that is, its value is entirely independent of the rate of change of its tonic motion. In eolotropic bodies K will have different values according to the direction in which it is taken; it is, however, always possible to choose our system of coordinates so that

$$T=\frac{1}{8\pi}(K_1P^2+K_2Q^2+K_3R^2),$$

where K_1, K_2, K_3 are the values of K in the direction of the so-called principal axes of the given body or crystal. We call T the tonic kinetic energy per unit-volume, or the density of the kinetic energy of the tonic motion. The quantities

$$\frac{K}{4\pi}\frac{dF}{dt},\quad \frac{K}{4\pi}\frac{dG}{dt},\quad \frac{K}{4\pi}\frac{dH}{dt}$$

are then the momenta, as defined by Lagrange, of the tonic motion per unit-volume; Maxwell denotes these momenta by f, g, h, putting

$$f=\frac{K}{4\pi}\frac{dF}{dt},\quad g=\frac{K}{4\pi}\frac{dG}{dt},\quad h=\frac{K}{4\pi}\frac{dH}{dt}.\ \ldots\ldots\ldots(3)$$

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* Compare Maxwell's Scientific Papers, vol. I., p. 476. In his later publications, Scientific Papers, vol. I., p. 555; Treatise, vol. II., p. 223, § 599, Maxwell puts

$$P=-\frac{dF}{dt},\quad Q=-\frac{dG}{dt},\quad R=-\frac{dH}{dt}.$$

Moreover, certain forces shall resist the electrotonic displacement; the creation of these forces necessitates an expenditure of energy or work; let us denote this work per unit-volume $d\tau$ by $Vd\tau$, V being the density of this potential tonic energy.

If F, G, H were the components of the displacement of a particle of an ordinary elastic body, V would be a homogeneous function of the second degree of the differentials of these quantities with regard to the coordinates, namely,

$$V=\frac{\lambda}{2}\left[\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right]^2+\mu\left[\left(\frac{dF}{dx}\right)^2+\left(\frac{dG}{dy}\right)^2+\left(\frac{dH}{dz}\right)^2\right]$$
$$+\frac{\mu}{2}\left[\left(\frac{dH}{dy}+\frac{dG}{dz}\right)^2+\left(\frac{dF}{dz}+\frac{dH}{dx}\right)^2+\left(\frac{dG}{dx}+\frac{dF}{dy}\right)^2\right];\ \ldots(4)$$

the coefficients λ and μ of this expression are given by the quantities

$$\lambda=\frac{n}{(m+n)(m-2n)},\quad \mu=\frac{1}{2(m+n)},$$

where m and n denote the longitudinal and cross-sectional protractions per unit-volume per unit-force; $\frac{1}{m}$ is the so-called modulus of elasticity; the elastic constants λ and μ were introduced by Lamé. The last three expressions in the smaller brackets are the so-called shears per unit-volume, produced by the shearing or tangential forces acting on that volume. We next assume that the potential tonic energy of the *ether*—let us denote it by V_ϵ—is likewise a homogeneous function of the second degree of the differentials of the quantities F, G, H, but of a somewhat different form from the above expression (4), namely,

$$V_\epsilon=\frac{\nu}{2}(a^2+b^2+c^2),\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $$a=\frac{dH}{dy}-\frac{dG}{dz},\quad b=\frac{dF}{dz}-\frac{dH}{dx},\quad c=\frac{dG}{dx}-\frac{dF}{dy};\ \ldots(6)$$

here ν is a constant belonging to the body in question, it corresponds to the constants λ and μ of expression (4); a, b, c are the so-called curls of the tonic vector (F, G, H).

However plain and simple the above assumption (5) may appear, it is by no means easy to form a definite mechanical conception of its meaning. If we conceive F, G, H as simply the components of the displacement of an ether particle, so that the tone of the ether corresponds to the elastic displacement of a particle of an ordinary solid, $K/4\pi$ would be its density, and a, b, c twice the rotations of any volume-element (more exactly twice its so-called mean rotations or the rotations of its diagonals) about three axes parallel to the coordinate-axes x, y, z respectively. In order to interpret expression (5) it would then be necessary to assume that a force proportional to this angular rotation resists it (rotation); for suppose an ether-element $dx\,dy\,dz$ turned through the angle a round the x-axis (from its initial position), the resisting force would then be proportional to a and $dx\,dy\,dz$, namely,

$$\nu a\,dx\,dy\,dz;$$

hence in increasing the angle a to $(a+da)$ the work done or the increment of the potential energy of this resisting force would be

$$\nu a\,dx\,dy\,dz\,da$$

with regard to the x-axis; the total potential energy with regard to this axis would therefore be

$$\int_0^a \nu a\,dx\,dy\,dz\,da = \frac{\nu}{2}a^2 dx\,dy\,dz;$$

and the total potential energy with regard to all three axes

$$\frac{\nu}{2}(a^2+b^2+c^2)dx\,dy\,dz,$$

which is our above expression (5) for V_ϵ. This is the

function which Lord Kelvin assumes for so-called quasi-rigid bodies such as the ether. He had evidently noticed that the above curls (6) were the only remaining expressions capable of physical interpretation that do not occur in the expression (4) for the potential energy V for ordinary elastic bodies; it may, therefore, be due to this alone that he conceived the idea of introducing these curls in forming an expression for V_ϵ. In assuming this simple expression for V_ϵ—V_ϵ is a function of a, b, c only—we should observe that we are ascribing the potential tonic energy to the rotation only of the volume-elements, and not to their elongation (protraction) or shear, as in the case of ordinary elastic bodies.

The above interpretation of F, G, H would incur several shortcomings. In the first place, there would necessarily be an influx or efflux of ether, that is, a change in its density $K/4\pi$, wherever so-called real electricity appeared, as we shall see later in § 6. The expression for the density of this real electricity is, according to formula (2) of Chapter III.,

$$\epsilon_r = \frac{1}{4\pi}\left[\frac{d(KP)}{dx}+\frac{d(KQ)}{dy}+\frac{d(KR)}{dz}\right]^*,$$

or, if we assume that K is constant with regard to x, y, z throughout the region in question,

$$\epsilon_r = \frac{K}{4\pi}\left[\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right].$$

From this expression it would now follow that, wherever

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) \gtrless 0$$

—this expression is the density of the so-called free electricity (cf. Chapter VI.),—the quantity K would be altered by the tonic motion, that is, it would no longer remain constant throughout the given body. This

* These constants, K and D, differ from those of formula (2) of Chapter III., as different systems of units have been employed.

variation of K might be attributed under certain conditions to the presence of the molecules of the body. On the other hand, if we consider K as constant with regard to the tonic motion, it would be just as difficult to account for the creation of free electricity; for in this case, where, namely, the ether is assumed to be incompressible, and hence its density $K/4\pi$ to be constant throughout, it would be necessary to introduce the condition

$$\delta(dx\, dy\, dz) = 0$$

into Hamilton's principle; this would now give rise to terms which correspond to the pressure in incompressible fluids, and which are entirely unknown to Maxwell's theory of electricity and magnetism. Such terms actually appear in von Helmholtz's theory of electrodynamics; their introduction seems, however, only to involve the mechanical meaning of his equations.

In order to afford the student a sound domain for research I offer here the few following suggestions, which seem to me to tend toward characterizing a motion quite similar to the tonic motion. Suppose every volume-element to contain a granule, and F, G, H to be the angles of rotation about three axes parallel to the coordinate axes, through which this granule has been turned, at any given time t. P, Q, R must then be interpreted as the components of the angular velocity, and $K/4\pi$ as the moment of inertia of the granule with regard to its instantaneous axis of rotation in order that

$$\frac{K d\tau}{8\pi}(P^2 + Q^2 + R^2)$$

may represent the kinetic energy of the granule. To find the kinetic energy of any particle m of the granule at the distance ρ from its instantaneous axis Δ of rotation, resolve its velocity about this axis into the three component velocities about the x, y, z-axes, as

indicated in the annexed figure; these velocities are evidently

$$\rho P, \ \rho Q, \ \rho R.$$

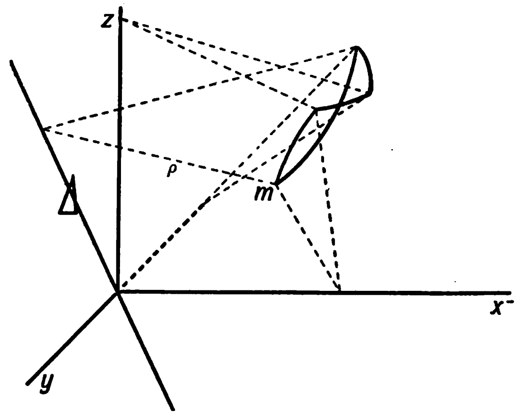

FIG. 1.

The kinetic energy of the particle m with regard to these axes will therefore be

$$\frac{m}{2}\rho^2 P^2, \ \frac{m}{2}\rho^2 Q^2, \ \frac{m}{2}\rho^2 R^2.$$

Hence its total kinetic energy

$$\frac{m}{2}\rho^2(P^2+Q^2+R^2),$$

and that of the granule

$$\Sigma\frac{m}{2}\rho^2(P^2+Q^2+R^2)=\tfrac{1}{2}(P^2+Q^2+R^2)\Sigma m\rho^2. \quad \text{[Q.E.D.}$$

For eolotropic bodies we should have to imagine the moment of inertia different along different axes, but, nevertheless, entirely independent of the configuration of the granule.

The interpretation of expression (5) presents even more serious difficulties than those just encountered. To

interpret this expression let us recall one of Maxwell's old conceptions, and assume that small particles are inserted between each pair of rotating granules and retard their rotations like friction-balls. If the average displacement of all these particles along the x-axis is proportional to a, and we assume that it gives rise to a force proportional to itself, the work done by these displacements along this axis will then be proportional to a^2, and similarly those along the y and z-axes proportional to b^2 and c^2 respectively. However forced the conception of such friction-balls may seem, it demonstrates at least that a motion can be conceived, which mechanically possesses all the properties that have been ascribed to the tonic motion. The desired resistance could also be effected by assuming a continual variation in the direction of the axis of rotation of the rotating granules, provided this assumption were consistent with conditions to be mentioned below. Every such assumption or hypothesis used to illustrate mechanically the tone or any of its properties we shall designate as a mechanical analogy or dynamical illustration.

We can regard the angles of rotation F, G, H as composed of two addenda, F_1, G_1, H_1 and F_2, G_2, H_2, and can conceive the former as infinitely large provided they only satisfy the conditions

$$\frac{dH_1}{dy}=\frac{dG_1}{dz},\quad \frac{dF_1}{dz}=\frac{dH_1}{dx},\quad \frac{dG_1}{dx}=\frac{dF_1}{dy}, \quad\text{..........(7)}$$

and the latter remain infinitely small, for then the expression for V,

$$V=\frac{\nu}{2}\left\{\left[\frac{d(H_1+H_2)}{dy}-\frac{d(G_1+G_2)}{dz}\right]^2\right.$$
$$\left.+\left[\frac{d(F_1+F_2)}{dz}-\frac{d(H_1+H_2)}{dx}\right]^2+\left[\frac{d(G_1+G_2)}{dx}-\frac{d(F_1+F_2)}{dy}\right]^2\right\}$$

will always remain finite. The mechanical interpretation of these conditions (7) can be shown to be that the

directions of the axes of rotation of the granules vary very slowly—gradual vibratory or cyclic variations in the directions of the axes of rotation, as those in the earth's axis, would be such. Consequently, in assuming the second interpretation of F, G, H as angles of rotation, we must exclude all rapid changes in the directions of the axes of rotation of the granules, and must thus abandon the second dynamical illustration for the forces a, b, c mentioned on the preceding page.

For reasons similar to the above the same conditions (7) must also hold for F, G, H, when interpreted as linear displacements with regard to the coordinate axes; here the mechanical meaning of these conditions evidently corresponds to the assumption of very small rotations.

The second mechanical interpretation of the expressions for T and V just offered bears a certain similarity to an old theory of Hankel * and a hypothesis of Sommerfeld †; the only difference is that the latter scientist does not assume friction molecules or balls, and thus encounters new difficulties.

It is, of course, possible that quite different motions, for example, vibratory or irregular zigzag motions such as the average motions of gas molecules, would lead to the same equations as the above. As, however, the treatment of such motions would be much more complicated and there is no real reason for any special choice, we shall not pursue this line of investigation further, but confine ourselves to an examination of the general properties of the tone as specified above.

If we accept any given mechanical analogy for our fundamental equations, we can imagine all quantities measured in the mechanical or natural units in which

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* Hankel, *Poggendorf's Annalen* 126, p. 440, 1865; 131, p. 607, 1867; cf. also Helm., *Wiedemann's Annalen* 47, p. 743, 1892.

† Sommerfeld, *Wiedemann's Annalen* 46, p. 139, 1892; Reiff's *Elasticität und Electricität*, Frieburg, akademischer Verlag., 1893.

they would naturally be measured in conformity to this analogy. We shall adopt this system of units for the present, indicating it by suffixing the index n to the quantities in question. We should thus write equation (2) as follows:

$$T=\frac{K}{8\pi}(P_n^2+Q_n^2+R_n^2), \qquad\qquad (8)$$

where the quantities with the index n are now supposed to be measured in their natural units.

For brevity let us introduce in place of ν the constant

$$\mu=\frac{1}{4\pi\nu},$$

and in place of a, b, c the quantities

$$\alpha=4\pi\nu a=\frac{a}{\mu}, \quad \beta=4\pi\nu b=\frac{b}{\mu}, \quad \gamma=4\pi\nu c=\frac{c}{\mu},$$

Equations (5) and (6) can then be written as follows:

$$V=\frac{\mu}{8\pi}(\alpha_n^2+\beta_n^2+\gamma_n^2), \qquad\qquad (9)$$

and

$$\mu\alpha_n=\frac{dH_n}{dy}-\frac{dG_n}{dz}, \quad \mu\beta_n=\frac{dF_n}{dz}-\frac{dH_n}{dx}, \quad \mu\gamma_n=\frac{dG_n}{dx}-\frac{dF_n}{dy} \qquad (10)$$

SECTION II. DERIVATION OF THE FUNDAMENTAL EQUATIONS.

We now return to our derivation of Maxwell's equations; this is effected by the application of Hamilton's principle, for, since we know the kinetic energy of, and the work done by, our volume-element $d\tau$, we can find by this principle the forces that tend to accelerate F, G, H or to increase P, Q, R. We write Hamilton's principle in the form

$$4\pi\iint dt\, d\tau(\delta T-\delta V)=0. \qquad\qquad (11)$$

Replacing here T and V by their above values, performing the indicated variations, and putting $d\tau = dx\,dy\,dz$, we have

$$\iiiint dt\,dx\,dy\,dz\Big[K\Big(P\frac{d\delta F}{dt}+Q\frac{d\delta G}{dt}+R\frac{d\delta H}{dt}\Big)$$
$$-\alpha\Big(\frac{d\delta H}{dy}-\frac{d\delta G}{dz}\Big)-\beta\Big(\frac{d\delta F}{dz}-\frac{d\delta H}{dx}\Big)-\gamma\Big(\frac{d\delta G}{dx}-\frac{d\delta F}{dy}\Big)\Big]=0\ldots(12)$$

Those terms which contain variations, where a differentiation with regard to the time is to be performed, must be differentiated by parts with regard to the time, and those containing variations, where a differentiation with regard to the coordinates is indicated, must be differentiated partially with regard to the same. The limits for the time and the values of the variables in their initial and final positions are regarded as constant in Hamilton's equation, that is, any quantity that is a function of these values only, undergoes no variation whatsoever. Hence, in integrating partially with regard to the time, those terms, which depend only upon the initial and final values of the time and not upon the path of integration, will vanish. Such a term will be found in performing the partial integration of the first integral of our equation (12); we have, namely,

$$\iiint\int_{t_0}^{t_1} dx\,dy\,dz\,dt\,KP\frac{d\delta F}{dt}$$
$$=\left|\iiint dx\,dy\,dz\,K\frac{dF}{dt}\,\delta F\right|_{t_0}^{t_1}-\iiint\int_{t_0}^{t_1} dx\,dy\,dz\,dt\,K\,\delta F\frac{d^2F}{dt^2};$$

the first term of the right-hand member of this equation will vanish, since F, in conformity to the above, undergoes no variation at the initial and final times t_0 and t_1 in any volume-element of the ether. In integrating by parts with regard to the coordinates we must pay special attention to the limiting conditions that hold for the surface of the body in question. Let us perform such a

partial integration, for example that of the seventh integral of equation (12); we find then

$$\iiiint dt\,dx\,dy\,dz\,\beta\frac{d\delta H}{dx}$$

$$=\left|\iiint dt\,dy\,dz\,\beta\delta H\right|-\iiiint dt\,dx\,dy\,dz\,\delta H\,\frac{d\beta}{dx}\,dx.$$

If the body in question were an ordinary elastic body and Hamilton's equation in the above form (12) were valid (cf. also p. 17), the value of the expression $|\beta\delta H|$ at any given time t, and along any straight line AA' parallel to the x-axis (see the annexed figure)—t, y,

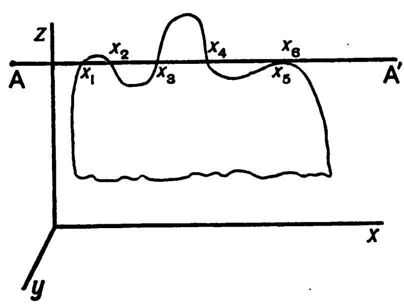

FIG. 2.

and z are then constant—would be found by subtracting the values of $\beta\delta H$ at the points x_2, x_4,... on the surface of the body, where this line AA' leaves it, from its values at the points x_1, x_3,... on its surface, where the same line enters it, and by taking the sum of these differences. Forming this expression for all lines AA' at any given time t and taking their sum we should have

$$\iint dy\,dz\left|\beta\delta H\right|=\left|\iint dy\,dz\,\beta\delta H\right|.$$

This expression integrated over any given period t_0 to t_1 would give then the above integral,

$$\left|\iiint dt\,dy\,dz\,\beta\delta H\right|, \quad\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

whose meaning thus becomes apparent.

The above variation δ is entirely arbitrary, except of course where $\delta=0$ in conformity to Hamilton's principle. Hence, if we should put $\delta=0$ at all points on the surface of the given body, expression (13) would vanish and no inaccuracy could arise from such an assumption; but this would not be the most general method of treatment. For the latter δ should be left arbitrary, and we should find the following surface-integrals from the partial differentiation of equation (12):

$$\left|\int dt\iint(\beta-\gamma)dy\,dz\,\delta H\right|, \quad \left|\int dt\iint(\gamma-a)dx\,dz\,\delta F\right|,$$

and

$$\left|\int dt\iint(a-\beta)dx\,dy\,\delta G\right|, \quad\ldots\ldots\ldots\ldots\ldots(14)$$

and from these three surface-integrals the following surface-conditions would then follow:

$$(\beta-\gamma)=0, \; (\gamma-a)=0, \text{ and } (a-\beta)=0. \ldots\ldots(15)$$

For ordinary elastic bodies these conditions or surface-equations are

$$U-\lambda X_x-\mu Y_x-\nu Z_x=0, \quad V-\mu Y_y-\nu Z_y-\lambda X_y=0,$$

$$W-\nu Z_z-\lambda X_z-\mu Y_z=0, \quad\ldots\ldots\ldots\ldots\ldots(16)$$

where U, V, W are the components of the force that act on the surface of the body at the point in question, X_x, $X_y \ldots Z_z$ the elastic forces (according to Kirchoff's notation) within the body and λ, μ, ν, the direction-cosines of the normal to the surface at that point.

As we have conceived so-called empty space to be filled with ether, we shall be obliged to regard the surfaces of the bodies in question as dividing-

surfaces between different media or masses of ether of different constitution. We could thus apply the same method as above for finding the surface-conditions to these dividing-surfaces, provided we only knew how they were constituted, that is, the forces that resided within them—these forces would correspond to the external forces U, V, W of formula (16), they do not appear in formulae (14) and (15), which were only developed for the sake of illustration; but as we know in fact nothing whatever about these forces or the dividing-surfaces themselves, our only alternative would be to resort to certain assumptions, in the choice of which much of an arbitrary character would be unavoidable. Even then we should have a considerable task before us; for as the integration can now no longer be limited to one body or mass of ether only, as in the theory of elasticity, but must be extended to the confines of ethereal space, we should have to determine the surface-conditions for each and every dividing-surface separately. We should moreover observe that, as we have accepted no definite mechanical analogy for our fundamental equations, the problem of finding the surface-conditions for adjoining media could not naturally be solved uniquely; at all events this would remain impossible, as long as we conceived the dividing-surfaces as purely mathematical surfaces.

Here we shall make the simplest assumption, that, namely, of which Maxwell has already made use and which has been further developed by von Helmholtz and Hertz.* We assume, namely, that adjoining bodies of different constitution are separated by very thin films and that within these films the properties of the one body are transformed very rapidly but continuously into those of the other; let us therefore designate these films as transition-films. For mixable substances, as water and a solution of water, zinc and mercury, or

* Cf. also Poincaré, *Electricité et optique*, vol. II., p. 11.

copper and melted zinc, such films could certainly be realized and might indeed be permanently created by transforming the bodies into the solid state, as by a lowering of their temperature, before the solution had advanced too far. Perhaps such transition-films could be created between any two bodies; or it may be that it is only the properties of the ether, to which alone our equations refer, that change continuously. In any case, whether mathematical discontinuities exist or not, or whether the transition-films belong to the bodies proper or are only to be associated with the ether itself, we shall assume that the conditions on the dividing-surfaces of different bodies or masses of ether are found by assuming continuous but very thin transition-films between them, within which our equations are also valid.

This assumption is without doubt the most probable and natural of the several assumptions that might still be made concerning dividing-surfaces. We gain by it at least a considerable mathematical advantage; for in assuming any other conditions we should be obliged in our partial differential with regard to the coordinates to investigate separately those terms, which refer to the dividing-surfaces, and to evaluate them in order to find the surface-conditions. In assuming the above transition-films, however, we do away with all discontinuities once for all, and are thus enabled to regard as the limits of our integrals the confines of space to which the electric and magnetic disturbances cannot of course extend; and in consequence of which all integrals dependent only upon these infinite limits, that is, all the surface-integrals, will vanish.

The above will still hold when the dividing-surfaces are real surfaces of discontinuity, provided only the same surface-conditions are assumed as those which result from the hypothesis of very rapid but continuous transitions.

In order to have a manner of signifying this deriv-

ation of the surface-conditions, we shall call it the principle of the continuity of transitions, including under this name the further hypothesis that neither the constants (K, μ, etc.) of the medium nor the densities of the kinetic T or the potential tonic energy V—provided, at least, no electromotive forces reside within the transition-films—can increase indefinitely, as the thickness of the film diminishes, since an infinite energy-density would necessitate the assumption of infinite forces at the point in question. We suppose, therefore, that adjoining bodies or media have a certain given constitution characterized by finite K, μ, T, V at every point of their transition-film. By equations (9) and (10) P, Q, R and a, b, c cannot then increase indefinitely as the thickness of the dividing-film diminishes, provided we make the further but most natural assumption that K and μ cannot be infinitely small. This is necessary, as otherwise these six quantities might become infinitely large, T and V still remain finite, as assumed above, and the validity of our equations remain undisturbed. The differential quotients taken at right angles to the film would not, however, necessarily remain finite as the thickness of the transition-film diminishes.

Since all the integrals, whose values depend only upon the limits of integration, vanish in conformity to the above assumptions, we find from equation (12) the following resulting equations after having performed all the partial integrations:

$$\iiiint dt\,dx\,dy\,dz\left\{\delta F\left[K\frac{dP}{dt}-\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right]\right.$$

$$\left.+\delta G\left[K\frac{dQ}{dt}-\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right]+\delta H\left[K\frac{dR}{dt}-\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right]\right\}=0.$$

As the variations δF, δG, δH are entirely independent of one another, and this equation must hold for all

variations, the following integrals must vanish identically:

$$\iiiint dt\,dx\,dy\,dz\,\delta F\left[K\frac{dP}{dt}-\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right],$$

$$\iiiint dt\,dx\,dy\,dz\,\delta G\left[K\frac{dQ}{dt}-\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right],$$

and

$$\iiiint dt\,dx\,dy\,dz\,\delta H\left[K\frac{dR}{dt}-\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right].$$

Moreover, as the variation δ is entirely arbitrary not only for every volume-element $d\tau$ but during every period dt—we except, of course, the initial and final positions of the volume-elements and the initial and final times (cf. p. 14)—the expressions under the integral-signs of these integrals must also vanish identically; for suppose we took any volume-element during any period, put the variations for all the other volume-elements during all the periods, and for this given element during all the other periods except this given period equal to zero; all the expressions under the integral signs of the above integrals, with the single exception of the expression for the given element during the given period, would then vanish; but according to our equation this only remaining expression must also vanish and for all variations δ; this is now only possible when its factor, the quantity in the brackets, vanishes, as maintained above. It follows, therefore, that

$$\frac{K}{4\pi}\frac{dP}{dt}=\frac{1}{4\pi}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right),\quad \frac{K}{4\pi}\frac{dQ}{dt}=\frac{1}{4\pi}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right),$$

$$\frac{K}{4\pi}\frac{dR}{dt}=\frac{1}{4\pi}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

We assume, of course, that the ponderable bodies themselves are at rest, that is, that K and μ do not vary with the time; they can, however, change their

values from point to point, that is, be functions of x, y, z. These equations (17) also hold for the electromagnetic state of ponderable bodies in motion, provided the latter carry the ether along with them without causing any disturbance in it (cf. § 40); such a case could only be approximately realized, for example, by the motion of a ponderable body charged with electricity, as an electrified meteoric mass, through outer space.

As according to the first interpretation of our fundamental expressions $K\frac{d\tau}{4\pi}$ plays the *rôle* of a mass and, according to our second view, that of an inertia, the right-hand members of equations (17) will represent the components of the force per unit-volume. These can be designated as the components of the tonic force acting on unit-volume. In addition to this force there shall now be a second one, which shall resist the tonic motion, and shall thus be called the resisting force or resistance. This resistance, similar to other resistances encountered by moving bodies—as bodies moving in our atmosphere—shall be opposite in direction and proportional to the tonic velocity, that is, equal to

$$-C\sqrt{P^2+Q^2+R^2},$$

where C is a constant, whose value can vary at different points of space—for bodies moving in our atmosphere the corresponding constant would be a function of its density, that is, a function of x, y, z. Its components along the coordinate-axes are then $-CP$, $-CQ$, $-CR$. This resistance is now in every respect analogous to friction; it must, therefore, give rise to a continual transformation of energy into heat. The force component $-CPd\tau$ acting for the time dt along the path dF would do the work $-CPdFd\tau$, that is, would diminish the energy of the volume-element by the quantity $CPdFd\tau$, or per unit-volume and per unit-time by the quantity $CP\frac{dF}{dt}$; similar expressions would

hold for the other two components. The energy transformed into heat per unit-volume and per unit-time with regard to all three axes would therefore be

$$W = C(PdF + QdG + RdH)/dt = C(P^2 + Q^2 + R^2),$$

$$\text{or} \quad W = C(P_n^2 + Q_n^2 + R_n^2). \qquad \text{(18)}$$

This is Joule's heat. On the other hand we shall assume that no energy is transformed into heat by the electrotonic forces, since they have a potential; this corresponds to neglecting the heat developed by magnetization or electric polarization in consequence of hysteresis.

Lastly, a term shall be added to the tonic force at every point of space, where hydro- or thermo-electromotive forces, or electromotive forces arising from friction, reside; and the only assumption we shall make concerning its form is that it shall be entirely independent of F_n, G_n, H_n. We shall denote the components of this unknown term per volume-element by

$$-CX_n d\tau, \quad -CY_n d\tau, \quad \text{and} \quad -CZ_n d\tau,\text{*}$$

and shall designate them as the components of the external electromotive force acting in that volume-element. We know, in fact, very little about X, Y, Z; we cannot even maintain that these quantities represent electromotive forces only, for they may even include quantities quite unknown to us. We can only assert that if X, Y, Z vanish everywhere, there can be no electromotive forces, but not conversely that, if there are no electromotive forces, X, Y, Z do not exist. In general the above components will represent a source of energy, thermal or chemical. The amount of energy

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* The constant C has been introduced here as factor for the sake of avoiding the appearance of another constant later (cp. equations (20)). We shall, however, for brevity refer to X, Y, Z as the electromotive forces.

derived from such a source per volume-element $d\tau$ during the time dt will then be

$$-C(X_n dF_n + Y_n dG_n + Z_n dH_n)d\tau,$$

or per unit-volume and per unit-time,

$$\begin{aligned}\Gamma &= -C(X_n dF_n + Y_n dG_n + Z_n dH_n)/dt \\ &= -C(X_n P_n + Y_n Q_n + Z_n H_n). \quad\text{..............(19)}\end{aligned}$$

By adding these new forces to our original expressions (17) we find the following complete equations:

$$\left.\begin{aligned}K\frac{dP_n}{dt} &= \frac{d\beta_n}{dz} - \frac{d\gamma_n}{dy} - 4\pi C(P_n + X_n) \\ K\frac{dQ_n}{dt} &= \frac{d\gamma_n}{dx} - \frac{d\alpha_n}{dz} - 4\pi C(Q_n + Y_n) \\ K\frac{dR_n}{dt} &= \frac{d\alpha_n}{dy} - \frac{d\beta_n}{dx} - 4\pi C(R_n + Z_n)\end{aligned}\right\} \quad\text{.........(20)}$$

Instead of introducing these new forces, the resistance and the external electromotive forces, after the examination of Hamilton's equation, it would have been more logical to have done so beforehand. Hamilton's equation would then have appeared in the complete form:

$$4\pi\iint dt\, d\tau(\delta T - \delta V - \delta W + \delta\Gamma) = 0, \quad\text{.........(21)}$$

where V, W, and $(-\Gamma)$ are the works done by the electrotonic forces, the resistance and the external electromotive forces respectively; and from this form of Hamilton's principle the complete equations (20) would have followed directly. For replace T, V, W, and Γ by their values (8), (9), (18), and (19), perform the indicated variations, and we obtain the following equation:

$$\iiiint dt\,dx\,dy\,dz\Big\{\frac{K}{4\pi}(P\delta P+Q\delta Q+R\delta R)$$

$$-\frac{\mu}{4\pi}(\alpha\delta\alpha+\beta\delta\beta+\gamma\delta\gamma)$$

$$-C\Big[(P\delta F+Q\delta G+R\delta H)+(X\delta F+Y\delta G+Z\delta H)\Big]\Big\}=0,$$

which integrated partially, according to exactly the same method as above and treated similarly, gives the desired equations (20).

The validity of equations (18) in those regions where $X=Y=Z=0$ cannot be doubted. On the other hand, the behaviour of those regions where external electromotive forces reside has been neither sufficiently investigated experimentally nor explained theoretically. If X, Y, Z behaved like mechanical forces, as the forces acting between bodies at rest, both equations (18) and (19) would then hold without any correction whatever. In most cases, however, motions seem to give rise to the external electromotive force, that is, external electromotive forces always seem to be preceded or, at least, accompanied by motions; as in hydro-cells first the dissociation and then the association of the chemical elements, and in thermo-currents the transmission of heat. We see, therefore, that in addition to the source of energy determined by equations (18) and (19) there can be another source of energy $\Lambda d\tau dt$ derived from the source that maintains the current, which is transformed directly into heat; in which case we should then have, in place of equation (18), the following expression for the heat generated:

$$W=\Lambda+C(P_n^2+Q_n^2+R_n^2), \qquad (22)$$

and in place of equation (19) the following expression for the energy spent:

$$\Gamma=\Lambda-C(X_nP_n+Y_nQ_n+Z_nR_n). \qquad (23)$$

We observe, however, that all the phenomena attri-

buted to Λ, Peltier's and Thomson's effects, deviations of Thomson's law for hydro-cells, etc., have not yet been entirely reconciled with theory. Where $X = Y = Z = 0$, that is, where no such source of energy Λ resides, no energy can be derived from this source directly and transformed into heat. In this case Λ of course vanishes and equations (18) are then empirically verified.

In order to have the same variables throughout it is well to introduce the quantities P, Q, R into equations (10); for this purpose we differentiate these equations with regard to t and have

$$\mu\frac{d\alpha_n}{dt}=\frac{dR_n}{dy}-\frac{dQ_n}{dz},\quad \mu\frac{d\beta_n}{dt}=\frac{dP_n}{dz}-\frac{dR_n}{dx},$$

$$\mu\frac{d\gamma_n}{dt}=\frac{dQ_n}{dx}-\frac{dP_n}{dy}. \quad\text{..................(24)}$$

Equations (8), (9), (20), and (24) are those from which we shall start in the following investigations; we shall, therefore, designate them as our fundamental equations. Equations (22) and (23), on the other hand, are to be regarded only as special equations or possible corrections to be applied to our fundamental equations, and will thus not be introduced in the ensuing more general investigations.

CHAPTER II.

SECTION III. OUR FUNDAMENTAL EQUATIONS CONSIDERED AS EMPIRICALLY GIVEN.

If we attach no importance to the above mechanical derivation of our fundamental equations (8, I.), (9, I.), (20, I.), and (24, I.) * we can have recourse to another method, that of Hertz; he simply writes down the six equations (20, I.) and (24, I.), observing that the strongest evidence of their validity lies in the fact that all phenomena duly follow from them and thus avoids all the hypotheses that were indispensable in the above derivation; a given definition of the quantities P, Q, R must then, however, be premised.

In the preceding chapter we have considered P, Q, R as the components of the velocity of an unknown but actually existing motion. Hertz, on the other hand, defines them as the components of the force which would act on the unit-quantity of electricity placed at the given point. Such a definition remains perfectly plausible as long as we adhere to the theory of action at a distance, since according to it quantities of electricity can be brought to or withdrawn from any point of the field without altering the electric state at its other points;

* In referring to formulae of other chapters I shall always insert the number of the Chapter directly after the formula in question; when no chapter is added the given Chapter is to be understood.

but according to Maxwell's theory this cannot be done without changing the state of the medium in the immediate neighbourhood of the given point.

Moreover, Hertz's definition cannot be applied even to the simplest case, namely, that of finding P, Q, R in the interior of a conductor traversed by an electric current; for as soon as the necessary hole has been bored in the conductor, the electro-motive forces within it have already ceased to act. If we bore the hole and thrust the unit-quantity of electricity into it so rapidly that the state of the immediate neighbourhood has not had sufficient time to become altered, we encounter the other difficulty, namely, that the electric action has not had time to be propagated across the hole, and Hertz's definition of P, Q, R thus becomes illusory.

Further, it must be proved that Hertz's definition of P, Q, R is consistent with the expression for the energy, since the forces which act on unit-quantity of electricity, and which, in conformity to this definition, should be P, Q, R, can be derived from this expression (cf. § 16).

We should, moreover, observe that Hertz defines the unit-quantity of electricity in one place by aid of P, Q, R, taking the expression

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)$$

as the definition for the density of the free electricity and the expression

$$\frac{1}{4\pi}\left[\frac{d(KP)}{dx}+\frac{d(KQ)}{dy}+\frac{d(KR)}{dz}\right]$$

as that for the density of the real electricity, and that in another place he makes use of the fundamental equations to prove that both of these expressions vanish not only throughout very extensive regions, but, in fact, throughout most all space. What can now be the meaning of defining P, Q, R in such regions as the

forces that would act on unit-quantity of electricity brought into these regions and placed at the given point, that is, as the forces that would act at that point if these expressions did not vanish, with other words, if these regions possessed entirely different properties from what they actually do possess? It would be hardly more satisfactory to take the other alternative and define P, Q, R as the forces that would act on vanishing unit-quantity of electricity. After Hertz had observed that these expressions generally vanish, he should have abandoned the above definition and identified the latter of the above expressions, in conformity to empirical laws, as the real electricity, having already accepted it as such in his definition. These criticisms are not intended to depreciate in any way the inestimable value of Hertz's representation of this subject, but only to indicate in what respects its logical keenness seems to lack completion.

Perhaps it would be sufficient to define P, Q, R as the components of a vector, giving rise to an energy

$$d\tau(T+V+W-\Gamma)$$

in every volume-element $d\tau$ of the medium. By replacing here T, V, W, Γ by their values (8, I.), (9, I.), (18, I.), and (19, I.) respectively, we should then find the following explicit expression for this energy:

$$d\tau\left\{\frac{K}{8\pi}(P^2+Q^2+R^2)+\frac{1}{8\pi\mu}\left[\left(\int_{-\infty}^{t}\frac{dR}{dy}dt-\int_{-\infty}^{t}\frac{dQ}{dz}dt\right)^2\right.\right.$$
$$\left.+\left(\int_{-\infty}^{t}\frac{dP}{dz}dt-\int_{-\infty}^{t}\frac{dR}{dx}dt\right)^2+\left(\int_{-\infty}^{t}\frac{dQ}{dx}dt-\int_{-\infty}^{t}\frac{dP}{dy}dt\right)^2\right]$$
$$\left.+C\Big[P(P+X)+Q(Q+Y)+R(R+Z)\Big]\right\}.$$

It was my intention to have regarded this definition or expression as the foundation of my investigations

in the preceding chapter, however obscured it might have there appeared; any such obscurity must be attributed to our more mechanical or illustrative treatment of this expression.

To define the magnetic forces a, b, c or α, β, γ as the forces that act on a magnetic pole of unit strength also seems to be entirely superfluous, for they are already defined by equations (6, I.) and (24, I.) as functions of P, Q, R, provided we only make the assumption that they were zero at some very remote period in the past, as $t = -\infty$. This assumption becomes necessary in order that we may express α, β, γ as functions of P, Q, R only; for integrate equations (24, I.) and find

$$\int_{-\infty}^{t} d\alpha = \alpha_t - \alpha_{-\infty} = \int_{-\infty}^{t} \left(\frac{dR}{dy} - \frac{dQ}{dz}\right) dt,$$

or

$$\alpha_t = \alpha_{-\infty} + \int_{-\infty}^{t} \left(\frac{dR}{dy} - \frac{dQ}{dz}\right) dt,$$

and similarly,

$$\beta_t = \beta_{-\infty} + \int_{-\infty}^{t} \left(\frac{dP}{dz} - \frac{dR}{dx}\right) dt, \quad \gamma_t = \gamma_{-\infty} + \int_{-\infty}^{t} \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dt,$$

from which it follows that $\alpha_t, \beta_t, \gamma_t$ would be functions of not only P, Q, R, but $\alpha_{-\infty}, \beta_{-\infty}, \gamma_{-\infty}$, unless the above assumption were made. We shall prove later (cp. § 33) that these magnetic forces are proportional to the forces that act on the pole of a so-called solenoid; we shall then accept Ampère's hypothesis of the existence of molecular currents and assume that the magnetic properties of iron magnets are due to their presence (cp. § 27).

SECTION IV. NATURAL SYSTEMS OF UNITS: THE ELECTROSTATIC SYSTEM.

If we accept a given mechanical analogy for our fundamental equations, the quantities of the preceding chapter will be defined mechanically, and it will thus be most convenient to measure them in their natural system of units, that is, in the system that corresponds to this analogy. According to our first interpretation, where, namely, F, G, H are linear displacements, P, Q, R will have the dimensions of a velocity and K that of a density, whereas a will have zero dimensions, that is, a will be an ordinary number (cf. formula (6, I.)); ν will therefore be the density of a potential energy (cf. formula (5, I.)), or, if we imagine potential energy converted into kinetic, a kinetic energy divided by a volume. Denoting the dimensions of length, mass, and time by l, m, and t respectively, we have then the following table:

$$[P_n]=lt^{-1},\quad [K]=ml^{-3},\quad [\nu]=\frac{ml^2t^{-2}}{l^3}=ml^{-1}t^{-2},$$

$$[\mu]=\left[\frac{1}{\nu}\right]=m^{-1}lt^2,\quad [a_n]=ml^{-1}t^{-2},\ \ldots\ldots\ldots\ldots(1)$$

where the brackets signify that only the dimensions of the quantities are referred to.

If we accept the second mechanical interpretation of our fundamental expressions, namely F, G, H, as the components of the angular rotation of our rotating granules, P, Q, R will have the dimensions of an angular velocity, K those of a moment of inertia, a, b, c those of inverse lengths, etc.; we can express this symbolically as above in the table,

$$[P_n]=t^{-1},\quad [K]=ml^2,\quad [a_n]=l^{-1},$$

$$[\nu]=ml^4t^{-2},\quad [a_n]=ml^3t^{-2}.\ \ldots\ldots\ldots\ldots\ldots(2)$$

As no mechanical analogy is empirically given, there can be no practical advantage gained in taking our measurements in any given system of natural units, since the dimensions themselves are different for different mechanical analogies (cf. Tables (1) and (2)). For practical purposes all systems of natural units have, therefore, been abandoned, and so-called standard units, that is, quantities accessible to experimental determination, have been introduced; such quantities are, for instance, the ratio of the K's or the μ's for different bodies. If we denote the values of K and μ for any body taken as standard body by K_a and μ_a, we can then determine experimentally the ratios

$$K : K_a = D, \quad \mu : \mu_a = M. \qquad (3)$$

Any body could be chosen as the standard body; it is, however, customary to choose the air as such. Other quantities that can be determined experimentally are the four energy-densities T, V, W, Γ. If we know the first two densities, we can then determine the six quantities

$$P_n\sqrt{K}, \ Q_n\sqrt{K}, \ R_n\sqrt{K}, \ \alpha_n\sqrt{K}, \ \beta_n\sqrt{K}, \ \gamma_n\sqrt{K},$$

directly from them, whereas the values of

$$C/K, \ X\sqrt{K}, \ Y\sqrt{K}, \ Z\sqrt{K}$$

follow from the determination of the densities W and Γ.

If we choose an insulator ($C=0$) as our standard body, and consider only those regions where external electromotive forces do not reside, that is, where

$$X = Y = Z = 0,$$

we can write our fundamental equations (2, I.) as follows:

$$K_a\frac{dP_n}{dt} = \frac{d\beta_n}{dz} - \frac{d\gamma_n}{dy}, \quad K_a\frac{dQ_n}{dt} = \frac{d\gamma_n}{dx} - \frac{d\alpha_n}{dz},$$

$$K_a\frac{dR_n}{dt} = \frac{d\alpha_n}{dy} - \frac{d\beta_n}{dx};$$

by formulae (1, I.) and (10, I.) and the condition that the expression

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0$$

—this condition denotes that the insulator has been at some previous time in the unelectrified state; it corresponds to condition (2, IV.), and is derived in a similar manner—these equations reduce to

$$\mu_a K_a \frac{d^2F}{dt^2}=\nabla^2 F, \quad \mu_a K_a \frac{d^2G}{dt^2}=\nabla^2 G, \quad \mu_a K_a \frac{d^2H}{dt^2}=\nabla^2 H,$$

where $$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.$$

From this familiar form of our equations it follows that

$$\frac{1}{\sqrt{\mu_a K_a}} \qquad (4)$$

is the velocity of propagation of the electric disturbances through the insulator. This velocity of propagation is, therefore, another quantity that can be determined experimentally. The quantity (4) could also be determined for a conductor, and the latter chosen as our standard body; not only its experimental determination would present even greater difficulties, but the velocity of propagation of the electric disturbances would no longer be given by this expression (4), but by a much more complicated one (see formula (17, IV.).

In the following we shall put

$$C/K_a=L, \quad \frac{1}{\sqrt{\mu_a K_a}}=\mathfrak{V}, \qquad (5)$$

where both L and $\mathfrak{V}$ are quantities that can be determined by experiment. An entirely new system of units, known as the electrostatic system of units, must then be employed for measuring the variables P, Q, R,

X, Y, Z, α, β, γ. As we shall adopt this system of units in the future, we shall drop the indices suffixed to the variables when the latter are measured in this system. The following relations will then evidently hold between the variables measured in any natural system of units and the same measured in the electrostatic system:

$$\left.\begin{array}{lll} P = P_n\sqrt{K_a}, & Q = Q_n\sqrt{K_a}, & R = R_n\sqrt{K_a} \\ X = X_n\sqrt{K_a}, & Y = Y_n\sqrt{K_a}, & Z = Z_n\sqrt{K_a} \\ \alpha = \alpha_n\sqrt{\mu_a}, & \beta = \beta_n\sqrt{\mu_a}, & \gamma = \gamma_n\sqrt{\mu_a} \end{array}\right\}, \quad \ldots\ldots(6)$$

where all the quantities can be determined experimentally.

If the dimensions of our variables are given in any natural system of units, we can find their dimensions in the electrostatic system by formulae (6); for example, the variables of our first natural system, that corresponding to the first mechanical interpretation of our fundamental equations, measured in the electrostatic system will have the dimensions

$$[(P_n)] = [(Q_n)] = [(R_n)] = [(X_n)] = [(Y_n)] = [(Z_n)]$$
$$= lt^{-1}\,.\,m^{\frac{1}{2}}l^{-\frac{3}{2}} = m^{\frac{1}{2}}l^{-\frac{1}{2}}t^{-1},$$
$$[(\alpha_n)] = [(\beta_n)] = [(\gamma_n)] = ml^{-1}t^{-2}\,.\,m^{-\frac{1}{2}}l^{\frac{1}{2}}t = m^{\frac{1}{2}}l^{-\frac{1}{2}}t^{-1},$$

and those of our second natural system, that corresponding to our second interpretation, the following dimensions:

$$[(P_n)] = \ldots = t^{-1}\,.\,m^{\frac{1}{2}}l = m^{\frac{1}{2}}lt^{-1},$$
$$[(\alpha_n)] = \ldots = ml^{3}t^{-2}\,.\,m^{-\frac{1}{2}}l^{-2}t = m^{\frac{1}{2}}lt^{-1}.$$

By replacing in the equations of the preceding chapter the quantities that appeared there measured in their natural units by the same quantities measured in electrostatic units we find the following system of equations,

which differ from those of the preceding chapter only in the values of the constants:

$$T=\frac{D}{8\pi}(P^2+Q^2+R^2), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$V=\frac{M}{8\pi}(\alpha^2+\beta^2+\gamma^2), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\left.\begin{aligned}\frac{D}{\mathfrak{V}}\frac{dP}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}-4\pi\frac{L}{\mathfrak{V}}(P+X)\\ \frac{D}{\mathfrak{V}}\frac{dQ}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}-4\pi\frac{L}{\mathfrak{V}}(Q+Y)\\ \frac{D}{\mathfrak{V}}\frac{dR}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}-4\pi\frac{L}{\mathfrak{V}}(R+Z)\end{aligned}\right\}, \quad\ldots\ldots\ldots (9)$$

$$\left.\begin{aligned}\frac{M}{\mathfrak{V}}\frac{d\alpha}{dt}=\frac{dR}{dy}-\frac{dQ}{dz},\quad \frac{M}{\mathfrak{V}}\frac{d\beta}{dt}=\frac{dP}{dz}-\frac{dR}{dx}\\ \frac{M}{\mathfrak{V}}\frac{d\gamma}{dt}=\frac{dQ}{dx}-\frac{dP}{dy}\end{aligned}\right\}, \quad\ldots\ldots\ldots (10)$$

$$\left.\begin{aligned}\frac{M}{\mathfrak{V}}\alpha=\frac{dH}{dy}-\frac{dG}{dz},\quad \frac{M}{\mathfrak{V}}\beta=\frac{dF}{dz}-\frac{dH}{dx}\\ \frac{M}{\mathfrak{V}}\gamma=\frac{dG}{dx}-\frac{dF}{dy}\end{aligned}\right\}, \quad\ldots\ldots\ldots (11)$$

where $F=F_n\sqrt{K_a}$, $G=G_n\sqrt{K_a}$, $H=H_n\sqrt{K_a}$,

and

$$W= L(P^2+Q^2+R^2), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$\Gamma=-L(XP+YQ+ZR). \quad\ldots\ldots\ldots\ldots\ldots (13)$$

The two special equations (22, I.) and (23, I.) then assume the following forms:

$$W=\Lambda+L(P^2+Q^2+R^2), \quad\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$\Gamma=\Lambda-L(XP+YQ+ZR). \quad\ldots\ldots\ldots\ldots\ldots (15)$$

Equations (7), (8), (9), (10) are our fundamental equations in this new or electrostatic system of units. As

this system is to be employed in the future, this new form of our fundamental equations will be used and referred to in the following. Although the electrostatic system is the only one used for magnetism, it is not always employed for electricity, for which there is another system in common use.

We should observe that if the constants of our standard body are not known, only the product KP_n^2 (for any body) can be determined from equation (8, I.); the quantity

$$C/K = C/K_a \,.\, K_a/K = \frac{L}{D}$$

can then be determined by formula (18, I.). The quantity L/D is a constant belonging to the given body; it is the so-called absorption of the electric disturbances or waves* (cf. also § 9). It is only when our standard body is also given that P_n and hence $K_aP_n^2$ of the given body are known in the electrostatic units of the standard body. The quantity $L = C/K_a$ can thus be determined for the given body, or we can determine its L directly from its constant L/D, since D is known.

SECTION V. CONDITIONS ON THE DIVIDING-SURFACES OF ADJOINING MEDIA.

The next question of interest is: What form do our fundamental equations (9) and (10) assume on the dividing-surfaces of adjoining bodies or media, or what are their surface-conditions? These can be derived without any difficulty from equations (9) and (10), and by the principle of the continuity of transitions, provided all infinite external electromotive forces are excluded from the transition-films.

* Cf. Cohn, *Berliner Berichte* 26, p. 405, 1889; also Hertz's *Gesammelte Abhandlungen*, p. 218.

For this purpose we construct an infinitely small rectangle with the sides ϵ and ζ on the given dividing-surface and conceive the surface-element $do = \epsilon\zeta$, in conformity to our principle of the continuity of transitions, not as a mathematical surface-element, but as an infinitely small parallelopiped with the base $\epsilon\zeta$ and the height δ

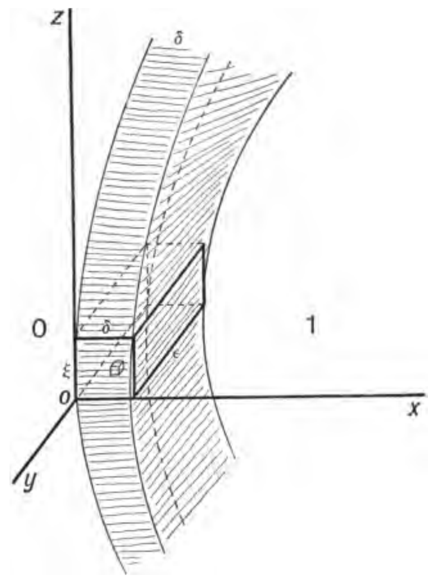

FIG. 3.

δ being the thickness of the transition-film at the point in question. For the ensuing derivation it is now necessary to make the assumption that δ is so small that lengths or quantities are conceivable that are large in comparison to δ and still so small that they can be considered as differentials, that is, that δ may be regarded as a differential of the second order. Analogous assumptions are made in all branches of mathematical physics, where volume-elements are constructed, that, although

large in comparison to the dimensions of a molecule, can still be considered as differentials themselves. Let ϵ and ζ be such lengths, which though large compared to δ are still infinitely small.

We take the origin of our system of coordinates in one corner of the parallelopiped $\delta\epsilon\zeta$, the abscissa-axis parallel to δ and the y and z-axes parallel to ϵ and ζ respectively. As equations (10) hold at every point (x, y, z) within the parallelopiped $\delta\epsilon\zeta$, we have

$$\frac{M}{\mathfrak{B}}\frac{d\alpha}{dt}=\frac{dR}{dy}-\frac{dQ}{dz},\quad \frac{M}{\mathfrak{B}}\frac{d\beta}{dt}=\frac{dP}{dz}-\frac{dR}{dx},\quad \frac{M}{\mathfrak{B}}\frac{d\gamma}{dt}=\frac{dQ}{dx}-\frac{dP}{dy}.$$

We now construct a second parallelopiped $dx\,dy\,dz$, that is infinitely small in comparison to $\delta\epsilon\zeta$, within the latter, multiply these equations, for example the second by $dx\,dy\,dz$, and integrate through $\delta\epsilon\zeta$; we have then

$$\int_0^\delta\int_0^\epsilon\int_0^\zeta \frac{M}{\mathfrak{B}}\frac{d\beta}{dt}dx\,dy\,dz-\int_0^\delta\int_0^\epsilon\int_0^\zeta dx\,dy\,dP+\int_0^\delta\int_0^\epsilon\int_0^\zeta dy\,dz\,dR=0$$

or

$$\int_0^\delta\int_0^\epsilon\int_0^\zeta \frac{M}{\mathfrak{B}}\frac{d\beta}{dt}\,dx\,dy\,dz-\int_0^\delta\int_0^\epsilon dx\,dy(P_1-P_0)$$
$$+\int_0^\epsilon\int_0^\zeta dy\,dz(R_1-R_1)=0. \quad\ldots\ldots\ldots(16)$$

The first integral of this equation can be written

$$\frac{1}{\mathfrak{B}}\,\overline{M\frac{d\beta}{dt}}\,\delta\epsilon\zeta,$$

where $\overline{M\dfrac{d\beta}{dt}}$ denotes the mean value of all $M\dfrac{d\beta}{dt}$. Similarly the second and third integrals can be written

$$(\overline{P}_1-\overline{P}_0)\delta\epsilon,\quad (\overline{R}_1-\overline{R}_0)\epsilon\zeta.$$

Equation (16) thus reduces to

$$\frac{1}{\mathfrak{V}}\overline{M\frac{d\beta}{dt}}\,\delta-(\overline{P}_1-\overline{P}_0)\frac{\delta}{\zeta}+(\overline{R}_1-\overline{R}_0)=0.$$

Since M, a, β, γ, P, Q, R are according to our hypothesis on p. 19 everywhere finite, and hence also their differentials with regard to the time, moreover, since δ is infinitely small in comparison to ϵ, the first two terms of this equation must be infinitely small in comparison to the members of the last term; it follows, therefore, that the last term must approximately vanish, that is, that

$$\overline{R}_1=\overline{R}_0;$$

$\overline{R}_1$ is the mean value of all the values that R assumes on the $x=\delta$ side of our parallelopiped $\delta\epsilon\zeta$, $\overline{R}_0$ its mean value on the side $x=0$. If the function R is continuous on the $x=\delta$ side of the given dividing-surface, we can write R_1 instead of $\overline{R}_1$, and similarly, R_0 instead of $\overline{R}_0$.

The third of the above equations (10) treated in a similar manner leads to the relation

$$Q_1=Q_0;$$

this relation can be found directly by interchanging the y and z-axes in the above. The first of equations (10) treated similarly leads to an identity. To find the special form assumed by equations (9) on dividing-surfaces we proceed exactly as above; we have

$$\frac{1}{\mathfrak{V}}\iiint\frac{d}{dt}(DQ)\,dx\,dy\,dz=\iint(\gamma_1-\gamma_0)\,dy\,dz-\iint(a_1-a_0)\,dx\,dy$$
$$-\frac{4\pi}{\mathfrak{V}}\iiint L(Q+Y)\,dx\,dy\,dz$$

or

$$\frac{1}{\mathfrak{V}}\overline{\frac{d}{dt}(DP)}\,\delta\epsilon\zeta=(\overline{\gamma}_1-\overline{\gamma}_0)\,\epsilon\zeta-(\overline{a}_1-\overline{a}_0)\,\delta\epsilon-\frac{4\pi}{\mathfrak{V}}\overline{L(Q+Y)}\,\delta\epsilon\zeta,$$

or retaining terms of the highest order of magnitude only,

$$(\bar{\gamma}_1-\bar{\gamma}_0)\epsilon\zeta=0,$$

hence
$$\gamma_1=\gamma_0.$$

The third of equations (9) similarly treated gives

$$\beta_1=\beta_0,$$

whereas the first leads to an identity.

To find relations between the P's and α's we first differentiate equations (9) and (10) respectively, the first with regard to x, the second to y, and the third to z, add and find

$$\iiint\frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]dx\,dy\,dz$$
$$+4\pi\iiint\frac{dL(P+X)}{dx}dx dy dz+4\pi\iiint\frac{dL(Q+Y)}{dy}dx dy dz$$
$$+4\pi\iiint\frac{dL(R+Z)}{dz}dx dy dz=0,$$

and
$$\frac{d}{dt}\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]=0 \text{ respectively.}$$

The evaluation of the second of these equations on any dividing-surface offers no difficulties; it is effected exactly as above, and leads to the conditional relation

$$\frac{d}{dt}(M_1\alpha_1-M_0\alpha_0)=0.$$

To evaluate the former equation we first examine the terms containing the external electromotive forces. These forces were introduced in the form CX, CY, CZ for the sake of avoiding the appearance of a third constant (see note, p. 22) in our fundamental equations; it would, however, have been more logical not to have made such use of this medium-constant, but to have

written simply X, Y, Z. For the following considerations this change of notation will be found desirable. The equation in question can then be written

$$\begin{aligned}\iiint \frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right] dx\,dy\,dz \\ +4\pi\iiint\left[\frac{d(LP)}{dx}+\frac{dX}{dx}\right] dx\,dy\,dz \\ +4\pi\iiint\left[\frac{d(LQ)}{dy}+\frac{dY}{dy}\right] dx\,dy\,dz \\ +4\pi\iiint\left[\frac{d(LR)}{dz}+\frac{dZ}{dz}\right] dx\,dy\,dz=0.\end{aligned}$$

We know now that the quantities P, Q, R and α, β, γ become discontinuous on the dividing-surfaces of adjoining bodies, which corresponds to rapid variations of these quantities in the transition-films; this will not, however, in general be true of the external electromotive forces X, Y, Z on the dividing-surfaces. If, for example, X remains continuous throughout the given transition-film, that is, if $\frac{dX}{dx}$ does not assume large values within it, $\int\frac{dX}{dx}dx$, integrated through the film, will vanish in comparison to $\int\frac{d(LP)}{dx}dx$. If, however, X is discontinuous on the given dividing-surface, that is, if $\frac{dX}{dx}$ assumes large values in the transition-film, $\int\frac{dX}{dx}dx$ will not vanish in comparison to $\int\frac{d(LP)}{dx}dx$. This latter case would perhaps be realized on the surface of contact between pure metals, as Zn and SO_4H_2, where a difference of potential,

$$\int\frac{dX}{dx}dx=\phi_{12},$$

is supposed to exist; in this case the above equation

treated similarly to equations (9) and (10) would assume the special form

$$\frac{d}{dt}(\overline{D_1P_1-D_0P_0})\epsilon\zeta+\frac{d}{dt}(\overline{D_1Q_1-D_0Q_0})\delta\zeta+\frac{d}{dt}(\overline{D_1R_1-D_0R_0})\delta\epsilon$$
$$+4\pi[(\overline{L_1P_1+X_1})-(\overline{L_0P_0+X_0})]\epsilon\zeta$$
$$+4\pi[(\overline{L_1Q_1+Y_1})-(\overline{L_0Q_0+Y_0})]\delta\zeta$$
$$+4\pi[(\overline{L_1R_1+Z_1})-(\overline{L_0R_0+Z_0})]\epsilon\zeta=0;$$

or, if we retain terms of only the first order of magnitude, that is, those without the factor δ,

$$\frac{1}{4\pi}\frac{d}{dt}(D_1P_1-D_0P_0)+L_1P_1+X_1-L_0P_0-X_0=0. \quad \ldots(17)$$

The equations in Q and R would then evidently have to be written

$$\left.\begin{aligned} Q_1-Q_0&=\frac{d\phi_{12}}{dy} \\ R_1-R_0&=\frac{d\phi_{12}}{dz}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(18)$$

The more common or general form of these equations would, however, be that, where X, Y, Z may be rejected, that is, where any potential difference due to the presence of external forces may be neglected; these equations then reduce to

$$\frac{1}{4\pi}\frac{d}{dt}(D_1P_1-D_0P_0)+L_1P_1-L_0P_0=0,$$
$$Q_1-Q_0=0, \quad R_1-R_0=0,$$

whereas the equations in α, β, γ remain unaltered, namely, ...(19)

$$\frac{d}{dt}(M_1\alpha_1-M_0\alpha_0)=0, \quad \beta_1-\beta_0=0, \quad \gamma_1-\gamma_0=0$$

The surface-conditions referred to any system of

coordinates, where, namely, the normal n to the dividing surface makes the angle (n, x), (n, y), (n, z) with the coordinate-axes, will evidently be the same as those referred to the above special system, provided we only conceive that each quantity is referred either to the tangential plane τ to the given dividing surface or to its normal n instead of to the yz-plane or the x-axis respectively of the above system. We should have then

$$\left.\begin{aligned} \frac{1}{4\pi}\frac{d}{dt}(D_1P_{1,n}-D_0P_{0,n})+L_1P_{1,n}-L_0P_{0,n}=0 \\ Q_{1,\tau}-Q_{0,\tau}=0, \quad R_{1,\tau}-R_{0,\tau}=0 \\ \frac{d}{dt}(M_1\alpha_{1,n}-M_0\alpha_{0,n})=0 \\ \beta_{1,\tau}-\beta_{0,\tau}=0, \quad \gamma_{1,\tau}-\gamma_{0,\tau}=0 \end{aligned}\right\}, \ldots\ldots (20)$$

where the index τ or n denotes that the quantity in question is referred to the given tangential plane or to its normal respectively. To express the quantities P_n, Q_τ, R_τ, α_n, β_τ, γ_τ in terms of P, Q, R, α, β, γ we resolve the vectors (P, Q, R) and (α, β, γ) along the plane τ and its normal n; we have then

$$\begin{aligned} P_n &= (P, Q, R)\cos[(P, Q, R), n] \\ &= (P, Q, R)\{\cos[(P, Q, R), x]\cos(n, x) \\ &\qquad + \cos[(P, Q, R), y]\cos(n, y) \\ &\qquad + \cos[(P, Q, R), z]\cos(n, z)\} \\ &= P_1\cos(n, x) + Q_1\cos(n, y) + R_1\cos(n, z), \end{aligned}$$

and similarly,

$$\left.\begin{aligned} Q_\tau &= P_1\cos(Q_\tau, x) + Q_1\cos(Q_\tau, y) + R_1\cos(Q_\tau, z) \\ R_\tau &= P_1\cos(R_\tau, x) + Q_1\cos(R_\tau, y) + R_1\cos(R_\tau, z) \\ \alpha_n &= \alpha_1\cos(n, x) + \beta_1\cos(n, y) + \gamma_1\cos(n, z) \\ \beta_\tau &= \alpha_1\cos(\beta_\tau, x) + \beta_1\cos(\beta_\tau, y) + \gamma_1\cos(\beta_\tau, z) \\ \gamma_\tau &= \alpha_1\cos(\gamma_\tau, x) + \beta_1\cos(\gamma_\tau, y) + \gamma_1\cos(\gamma_\tau, z) \end{aligned}\right\}, \ldots (21)$$

where Q_τ and β_τ and R_τ and γ_τ respectively refer to the same directions.

The following familiar conditions must now exist between these cosines:

$$\left.\begin{array}{l}\cos(Q_\tau, n)=\cos(Q_\tau, x)\cos(n,x)+\cos(Q_\tau,y)\cos(n,y)\\ \qquad\qquad +\cos(Q_\tau, z)\cos(n,z)=0\\ \cos(R_\tau, n)=\cos(R_\tau, x)\cos(n,x)+\cos(R_\tau,y)\cos(n,y)\\ \qquad\qquad +\cos(R_\tau, z)\cos(n,z)=0\\ \text{and}\quad \cos(Q_\tau, R_\tau)=0,\ \cos(\beta_\tau, n)=0,\ \cos(\gamma_\tau, n)=0\\ \text{and}\qquad\qquad \cos(\beta_\tau, \gamma_\tau)=0\end{array}\right\}\ldots(22)$$

The relation $Q_{1,\tau}-Q_{0,\tau}=0$ transformed to the x, y, z system of coordinates thus becomes

$$\left.\begin{array}{l}Q_{1,\tau}-Q_{0,\tau}=(P_1-P_0)\cos(Q_\tau, x)+(Q_1-Q_0)\cos(Q_\tau, y)\\ \qquad\qquad +(R_1-R_0)\cos(Q_\tau, z)\end{array}\right\}. \quad (23)$$

Comparing this relation with the first of conditions (22) we observe that the following proportion will satisfy both:

$$\frac{P_1-P_0}{\cos(n,x)}=\frac{Q_1-Q_0}{\cos(n,y)}=\frac{R_1-R_0}{\cos(n,z)};\ \ldots\ldots\ldots\ldots(24)$$

the relation $R_{1,\tau}-R_{0,\tau}=0$ leads to the same proportion; this proportion can also be obtained by choosing the two arbitrary coordinate axes of the tangential plane τ in such a manner that one of the three angles between the direction Q_τ and the axes x, y, z respectively becomes a right angle. This choice does not of course affect in any way the generality of our equations. For example, put the angle $(Q_\tau, z)=90°$; $\cos(Q_\tau, z)$ will then vanish, and the two equations in question will reduce to

$$(P_1-P_0)\cos(Q_\tau, x)+(Q_1-Q_0)\cos(Q_\tau, y)=0$$

and $$\cos(n,x)\cos(Q_\tau, x)+\cos(n,y)\cos(Q_\tau, y)=0,$$

which give $$\frac{P_1 - P_0}{\cos(n, x)} = \frac{Q_1 - Q_0}{\cos(n, y)}, \qquad\qquad (25)$$

and the third term of proportion (24) will then follow directly from the general equation (23), the first of conditions (22) and this relation (25).

Similarly, $\beta_{1,\tau} - \beta_{0,\tau} = 0$ and $\gamma_{1,\tau} - \gamma_{0,\tau} = 0$ referred to any system of coordinates will give the proportion

$$\frac{a_1 - a_0}{\cos(n, x)} = \frac{\beta_1 - \beta_0}{\cos(n, y)} = \frac{\gamma_1 - \gamma_0}{\cos(n, z)}. \qquad\qquad (26)$$

To transform the surface-conditions in P_n and a_n to any system of coordinates we replace P_n and a_n by their values (19), and find

$$\left.\begin{aligned}
&\left[\frac{1}{4\pi}\frac{d}{dt}(D_1P_1 - D_0P_0) + L_1P_1 - L_0P_0\right]\cos(n, x)\\
&+\left[\frac{1}{4\pi}\frac{d}{dt}(D_1Q_1 - D_0Q_0) + L_1Q_1 - L_0Q_0\right]\cos(n, y)\\
&+\left[\frac{1}{4\pi}\frac{d}{dt}(D_1R_1 - D_0R_0) + L_1R_1 - L_0R_0\right]\cos(n, z) = 0\\
&\text{and}\\
&\frac{d}{dt}\Big[(M_1a_1 - M_0a_0)\cos(n, x) + (M_1\beta_1 - M_0\beta_0)\cos(n, y)\\
&\qquad + (M_1\gamma_1 - M_0\gamma_0)\cos(n, z)\Big] = 0
\end{aligned}\right\} \qquad (27)$$

CHAPTER III.

SECTION VI. CONCEPTION OF THE REAL AND NEUTRAL ELECTRICITIES. A CONCRETE REPRESENTATION AS A METHOD OF ILLUSTRATING THE MEANING OF THE INTEGRALS OF OUR EQUATIONS; ITS FIRST FEATURE.

WITH the exception of the assumptions which we shall require for the explanation of electro-magnetism, we shall not be obliged to make in the future any others than those already introduced. The contents of what follows will be only consequences of the assumptions already made, that is, results derived from the equations to which the assumptions themselves have led. To illustrate these consequences we shall often introduce new mechanical representations; we remark, however, here at the outset that the reader should not interchange this so-called concrete representation, which serves only as an illustration, with the mechanical foundation of the theory itself. Whoever insists upon associating only one analogy with a mechanical theory will necessarily only perceive one manner of representation in the preceding chapters; nevertheless, the difference does exist between the theory and the representations that serve to illustrate its several consequences.

A body, within which the value of L is so small that we may put L approximately equal to zero, is called an insulator (as glass). If now no external

electromotive forces are active in the given insulator at the time in question, X, Y, Z would surely not attain to such a degree of infinity during the next succeeding period dt that the products LX, LY, LZ would prove different from zero; in this case, $L=X=Y=Z=0$ initially, we should find the following equation by the differentiation of equations (9, II.), the first with regard to x, the second to y, and the third to z, and their addition under the assumption that D is independent of t, that is, that the ponderable bodies themselves are at rest, and hence remain unaltered as far as their ponderable properties are concerned:

$$\frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]=0.$$

The expression in the large brackets cannot, therefore, vary with the time. If its initial value is zero it will remain equal to zero, as long as no external electromotive forces are brought to act at the given point; if, on the other hand, it has already assumed a given value due to the previous action of external electromotive forces, it will retain that value during all future periods, provided it is neither annulled nor altered by the subsequent action of external forces. We have therefore

$$\frac{d}{dx}(DP)+\frac{d}{dy}(DQ)+\frac{d}{dz}(DR)=\text{const.}(t)=\mathrm{f}(x, y, z).$$

We next imagine that the external electromotive forces have the effect of filling the insulator with a fluid, whose density ϵ_r at every point is given by the expression

$$\epsilon_r=\frac{1}{4\pi}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]=\frac{1}{4\pi}\,\mathrm{f}(x, y, z), \ldots .(1)$$

but that this fluid is bound within it. $\epsilon_r d\tau$ will then be the constant quantity—constant with regard to t—

of fluid contained in the given volume-element $d\tau$. We shall designate $\epsilon_r d\tau$ as the real electric fluid or electricity contained in $d\tau$ and ϵ_r as its density. Since the expression (1) for ϵ_r can assume negative as well as positive values, it is customary to imagine that initially, in the normal unelectrified state, each volume-element $d\tau$ contained a quantity $m d\tau$ of this fluid, which we shall call its neutral electric fluid or electricity, and that $\epsilon d\tau$ is only the excess or deficit of neutral electricity in $d\tau$; m is the density of the neutral electricity. When $\epsilon d\tau$ is positive, it is often referred to as the redundant fluid, and, when negative, as the deficient fluid. The neutral electricity is of course also to be conceived as bound within the insulator, as long as no external electromotive forces act.

The above representation for ϵ is known as the unitary or one-fluid theory of electricity. It not only does not suffice in all cases, but contains several weak points (cf. sec. XVI). Although it is the simpler of the two theories, we shall adopt here the more complicated dualistic or two-fluid theory of electricity, since the number of difficulties seems to be reduced to a minimum. As, however, we are not operating with real quantities, but only with fictitious ones, any preference can only be a matter of taste; we prefer, however, a greater complication of ideas to the slightest avoidable obscurity (cf. sec. XVI).

According to the above conceptions we shall thus consider electricity only as something (fluid) conceived by us for our own convenience to serve as an illustration of the meaning of the integrals of certain equations—equation (1) is such an equation—in strict contrast to the ether, which we shall regard as something really existing or material. These two conceptions are so confounded by Maxwell in his treatise that it is only with the greatest difficulty that the student is able to detect them distinctively. Although there is little doubt but that Maxwell attempted to draw a distinction

between these two conceptions, there is also sufficient proof that they were hardly so matured in his mind as to allow of an explicit exposition.

According to the dualistic theory there are now two electric fluids, the positive and the negative. A quantity of the former is always denoted by prefixing the positive sign, and one of the latter by prefixing the negative. In the neutral or unelectrified state each volume-element $d\tau$ is supposed to contain the quantity

$$\frac{md\tau}{2} \text{ of positive,}$$

and

$$-\frac{md\tau}{2} \text{ of negative fluid.}$$

When any volume-element $d\tau$ contains real electricity, it is supposed to have the quantities

$$\frac{m+\epsilon}{2}d\tau \text{ of positive,}$$

and

$$-\frac{m-\epsilon}{2}d\tau \text{ of negative fluid.}$$

The so-called neutral electricity is then the sum of the absolute values of these two quantities (without regard to sign), namely,

$$\left(\frac{m+\epsilon}{2}+\frac{m-\epsilon}{2}\right)d\tau = md\tau;$$

whereas the real electricity is their algebraic sum

$$\left(\frac{m+\epsilon}{2}-\frac{m-\epsilon}{2}\right)d\tau = \epsilon d\tau.$$

In the ideal non-conductor the neutral and real electricities are supposed to be bound.

The value of our concrete representation becomes more apparent, as soon as we proceed to the case, where L, X, Y, Z do not vanish. Here the partial differentiation of equations (9, II.), the first with regard to x, the

second to y, and the third to z, and their addition leads to the equation

$$\frac{d\epsilon_r}{dt}+\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}=0. \quad \ldots(2)$$

The general integration of this equation is not only difficult but laborious for even the simplest conditions or assumptions; it is therefore a happy circumstance that its meaning can be interpreted comparatively easily by our concrete representation. Its first feature is the following: we conceive the electric fluids as flowing through the conductor, the positive with a velocity, whose components along the coordinate axes are

$$u'=\frac{L}{m}(P+X), \quad v'=\frac{L}{m}(Q+Y), \quad w'=\frac{L}{m}(R+Z), \ldots(3)$$

and the negative with a velocity equal but opposite in direction to that of the positive fluid. The quantity of positive fluid that passes in through any surface-element do in the interior of the conductor from its one side s_1 to its other, s_2, during the time dt will then be

$$\left(\frac{m}{2}+\frac{\epsilon}{2}\right)\frac{L}{m}[(P+X)\cos(n, x)+(Q+Y)\cos(n, y)$$
$$+(R+Z)\cos(n, z)]dt\,do,$$

where n denotes the normal to do drawn from s_1 towards s_2. The expressions

$$P\cos(n, x)+Q\cos(n, y)+R\cos(n, z),$$
$$X\cos(n, x)+Y\cos(n, y)+R\cos(n, z), \ldots\ldots\ldots(4)$$

are the components of the vectors (P, Q, R) and (X, Y, Z) respectively along the normal n. Denoting them by N and S respectively we can write the above expression as follows:

$$\frac{m+\epsilon}{2m}L(N+S)dt\,do. \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Similarly the quantity of negative fluid that passes through the same surface-element do in the opposite

direction (from s_2 towards s_1) during the same time will be

$$-\frac{m-\epsilon}{2m}L(N+S)dt\,do.$$

The algebraic sum of all the fluids that pass through do in the direction of the normal during the time dt will be therefore

$$L(N+S)dt\,do = \omega\,dt\,do. \quad\text{..................}(6)$$

If n coincides with the direction of flow of the fluids, ω evidently becomes their so-called total current strength; we shall then denote it by Ω and write

$$\Omega = \sqrt{p^2+q^2+r^2};$$

its components p, q, r along the coordinate axes are evidently

$$\left.\begin{aligned} p &= L(P+X) = mu' \\ q &= L(Q+Y) = mv' \\ r &= L(R+Z) = mw' \end{aligned}\right\}, \quad\text{..................}(7)$$

We next construct within the conductor a parallelopiped $dx\,dy\,dz$, and determine the quantity of positive fluid that flows in through the side $dy\,dz$ opposite the negative abscissa-axis during the time dt; it is evidently given by the expression

$$\frac{m+\epsilon}{2}\frac{L}{m}(P+X)dy\,dz\,dt;$$

similarly the quantity of negative fluid that flows out through the same side during the same time will be

$$-\frac{m-\epsilon}{2}\frac{L}{m}(P+X)dy\,dz\,dt.$$

The algebraic sum of all fluids flowing into the parallelopiped through this side—fluids flowing out through it are of course to be considered as negative fluids flowing into it—will therefore be

$$L(P+X)dy\,dz\,dt.$$

The algebraic sum of all fluids flowing out through its opposite side $dy\,dz$ must now be given by the analogous expression

$$L(x+dx,\, y,\, z)[P(x+dx,\, y,\, z)+X(x+dx,\, y,\, z)]\,dy\,dz\,dt,$$

which developed according to Taylor's theorem gives

$$L(P+X)\,dy\,dz\,dt+\frac{dL(P+X)}{dx}\,dx\,dy\,dz\,dt.$$

Similar expressions hold for its two other pairs of sides ($dz\,dx$ and $dx\,dy$). The algebraic sum of all the fluids that have passed out of the parallelopiped and those that have entered it during the time dt will be then

$$dx\,dy\,dz\,dt\left[\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}\right].$$

Here every quantity of negative fluid entering or positive leaving the parallelopiped has been reckoned positive, whereas every quantity of negative fluid entering or positive leaving it has been given the negative sign. This expression must now represent the deficit $(-d\epsilon\,d\tau)$ of the real electricity (algebraic sum of all the fluids) in $d\tau$ during the time dt; the following equality must therefore exist:

$$\left[\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}\right]d\tau\,dt=-d\epsilon\,d\tau,$$

which we recognize as equation (2) above. The conception of such a fluid or fluids thus offers a means for illustrating and comprehending the real meaning not only of equation (2) but of the occurrent phenomena expressed by it; it enables us figuratively speaking to integrate this equation.

The sum of the absolute values of all fluids, that is, the neutral electricity, contained in $d\tau$ evidently increases during the time dt by the amount

$$\frac{1}{m}d\tau\,dt\left[\frac{d\epsilon L(P+X)}{dx}+\frac{d\epsilon L(Q+Y)}{dy}+\frac{d\epsilon L(R+Z)}{dz}\right].$$

As we always assume m to be large in comparison with ϵ, this expression will be small in comparison with the initial quantity of neutral electricity $md\tau$ contained in $d\tau$; in order to render the above representation complete in every respect we should indeed be obliged to assume that these small variations in the neutral electricity disappear so rapidly that they do not give rise to any phenomena susceptible to observation. We shall see later (cf. § 18) that in the interior of conductors, where alone any appreciable flow is possible, ϵ assumes a value different from zero for only the shortest period; consequently one of the factors L or ϵ will always approximately vanish, and this accumulation of neutral electricity can thus be entirely overlooked in the observation of ordinary phenomena.

SECTION VII. EXPRESSIONS FOR ϵ_r AND $\frac{d\epsilon_r}{dt}$ ON THE DIVIDING-SURFACES OF ADJOINING MEDIA.

For the purpose of finding the surface-conditions for ϵ_r and $\frac{d\epsilon_r}{dt}$ on dividing-surfaces, we proceed exactly as in § 5; we conceive, namely, the given dividing-surface to have the thickness δ, and integrate the expression in question over δ—we assume of course that the given expression holds at every point of the transition-film. Multiplying expression (1) for ϵ_r by $do\,dx$, an element of the transition-film, and integrating from $x=0$ to $x=\delta$, —let the system of coordinates be chosen as in § 5, the normal to the surface coinciding with the x-axis—we have

$$do\int\epsilon\,dx=\frac{do}{4\pi}\int\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]dx;$$

$do\int\epsilon\,dx$ is the quantity of real electricity contained in the volume-element δdo, that is, the quantity of real

electricity that resides on the dividing-surface-element do of the two bodies; denoting this quantity by $E\,do$, and performing the indicated integration, we find

$$E\,do = \frac{do}{4\pi}\left\{(D_1P_1 - D_0P_0) + \left[\frac{d(D_1Q_1 - D_0Q_0)}{dy} + \frac{d(D_1R_1 - D_0R_0)}{dz}\right]\delta\right\}.$$

Retaining here terms of a higher order of magnitude only (cf. p. 41), we have

$$E = \frac{1}{4\pi}(D_1P_1 - D_0P_0). \qquad \text{(8)}$$

If the normal to the dividing-surface does not coincide with the x-axis, the expression for E will evidently have the form

$$E = \frac{1}{4\pi}(D_1N_1 - D_0N_0),$$

where N is the component of the vector (P, Q, R) along the normal to the surface. This expression transformed to the x, y, z system of coordinates gives (cf. § 5)

$$E = \frac{1}{4\pi}[(D_1P_1 - D_0P_0)\cos(n, x) + (D_1Q_1 - D_0Q_0)\cos(n, y) + (D_1R_1 - D_0R_0)\cos(n, z)].$$

The equation for $\frac{d\epsilon}{dt}$ on any dividing-surface is obtained from equation (2) in a similar manner; we find

$$\frac{dE}{dt} = L_0(P_0 + X_0) - L_1(P_1 + X_1), \qquad \text{(9)}$$

where X_0 and X_1 are to be put equal to zero, when $\phi_0 = \phi_1$ (cf. p. 42). If we replace E by its value (8), we have

$$\frac{1}{4\pi}\frac{d}{dt}(D_1P_1 - D_0P_0) + L_1(P_1 + X_1) - L_0(P_0 + X_0) = 0,$$

which is equation (17) of § 5. The general form of this equation is to be found in the same article.

We should now be able to derive equations (8) and (9) directly from our concrete representation, since other-

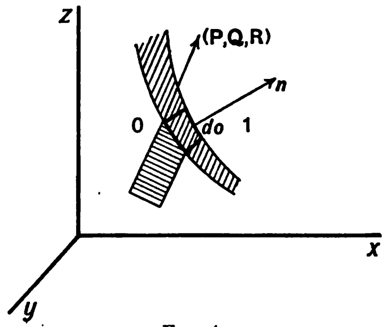

FIG. 4.

wise such a representation would be valueless. For this purpose we refer the given dividing-surface to any rectangular system of coordinates x, y, z, and determine the quantities of positive and negative fluid that pass through any one of its surface-element *do* during the time *dt*. The quantity of positive fluid entering the transition-film from left to right (in the annexed figure) through this surface-element *do* will evidently be

$$\frac{m+\epsilon}{2}\sqrt{u'^2+v'^2+w'^2}\cos(n, \sqrt{u'^2+v'^2+w'^2})dt\,do,$$

and the quantity of negative fluid leaving it from right to left

$$-\frac{m-\epsilon}{2}\sqrt{u'^2+v'^2+w'^2}\cos(n, \sqrt{u'^2+v'^2+w'^2})dt\,do.$$

The expression

$$\sqrt{u'^2+v'^2+w'^2}\cos(n, \sqrt{u'^2+v'^2+w'^2})$$

can be written as follows:

$$\begin{aligned}\sqrt{u'^2+v'^2+w'^2}\{&\cos(n, x)\cos(\sqrt{u'^2+v'^2+w'^2}, x)\\ &+\cos(n, y)\cos(\sqrt{u'^2+v'^2+w'^2}, y)\\ &+\cos(n, z)\cos(\sqrt{u'^2+v'^2+w'^2}, z);\end{aligned}$$

or, since

$$\cos(\sqrt{u'^2+v'^2+w'^2}, x) = \frac{u'}{\sqrt{u'^2+v'^2+w'^2}}, \text{ etc.,}$$

as follows:

$$u'\cos(n, x) + v'\cos(n, y) + w'\cos(n, z);$$

or, if we replace u', v', w' by their values (3), as follows:

$$\frac{L}{m}[(P+X)\cos(n, x) + (Q+Y)\cos(n, y) + (R+Z)\cos(n, z)].$$

We can, therefore, write the given quantities of fluid as follows:

$$\frac{m+\epsilon}{2m} L\{(P+X)\cos(n, x) + (Q+Y)\cos(n, y) + (R+Z)\cos(n, z)\} dt\, do,$$

and

$$\frac{m-\epsilon}{2m} L\{(P+X)\cos(n, x) + (Q+Y)\cos(n, y) + (R+Z)\cos(n, z)\} dt\, do;$$

the sign of the second expression has become plus, since $\cos(n, \sqrt{u'^2+v'^2+w'^2})$ is negative for the negative fluid leaving the film from right to left.

The algebraic sum of these two quantities of fluid, that is, the amount of real electricity that enters the left surface-element *do* of the transition-film, will be

$$L_0[(P_0+X_0)\cos(n, x) + (Q_0+Y_0)\cos(n, y) + (R_0+Z_0)\cos(n, z)] dt\, do,$$

where the index 0 refers to the left side of the given film and the index 1 to its right or opposite side. Similarly the quantity of real electricity that leaves the transition-film through its opposite surface-element *do* will be

$$L_1[(P_1+X_1)\cos(n, x) + (Q_1+Y_1)\cos(n, y) + (R_1+Z_1)\cos(n, z)].$$

The deficit of the real electricity residing on any surface-element do of the dividing-surface between two bodies during the time dt, that is, the quantity

$$\left(-\frac{dE}{dt}\,dt\,do\right),$$

will therefore be

$$\begin{aligned}\{L_1[(P_1+X_1)\cos(n,x)+(Q_1+Y_1)\cos(n,y)&\\ +(R_1+Z_1)\cos(n,z)]&\\ -L_0[(P_0+X_0)\cos(n,x)+(Q_0+Y_0)\cos(n,y)&\\ +(R_0+Z_0)\cos(n,z)]\}\,dt\,do&\\ =-\frac{dE}{dt}\,dt\,do,&\end{aligned}$$

which gives

$$\begin{aligned}\frac{dE}{dt}=[L_0(P_0+X_0)-L_1(P_1+X_1)]\cos(n,x)&\\ +[L_0(Q_0+Y_0)-L_1(Q_1+Y_1)]\cos(n,y)&\\ +[L_0(R_0+Z_0)-L_1(R_1+Z_1)]\cos(n,z),&\end{aligned}$$

the desired equation (cf. equations (9) and (27, II.)), obtained directly from our concrete representation. This confirmation was in fact quite superfluous, since, according to our assumptions, the transition-film behaves exactly as the medium itself, that is, since all quantities vary continuously within it, and the same equations hold at every point of it that hold for the medium. Consequently, if our concrete representation leads to equations that hold for the interior of bodies, it will also give us the correct equations for their dividing-surfaces, and the confirmation of its validity for the transition-films thus becomes superfluous. The object of the above derivation was indeed chiefly to render the reader more familiar with the concrete representation itself. Henceforth we shall omit similar investigations.

SECTION VIII. SECOND FEATURE OF OUR CONCRETE REPRESENTATION.

We can represent the fictitious motion of the positive and negative fluids of the preceding article by conceiving that it is produced by external forces. The simplest assumption is that these fluids move with great friction, and that the friction is proportional to their velocity, the latter being in turn proportional to the force acting on the fluids. If K is the force that acts on unit-quantity of positive fluid, and if ξ, η, ζ are its components along the coordinate-axes, it is then evident that these components must be put proportional to the quantities $P+X$, $Q+Y$, $R+Z$, namely,

$$\xi=B(P+X), \quad \eta=B(Q+Y), \quad \zeta=B(R+Z).$$

It follows from a comparison of these expressions with formulae (3) that the factor of proportionality between the velocity of the given fluids and the force acting on their unit quantity is equal to

$$\frac{L}{mB},$$

and hence that

$$u'=\frac{L}{mB}\xi, \quad v'=\frac{L}{mB}\eta, \quad w'=\frac{L}{mB}\zeta.$$

For the purpose of determining B we must compare the heat developed by the friction of the fluids with that actually generated (Joule's heat). We cannot of course foresee whether it will be possible to reconcile the work done by these moving fluids and transformed into heat with that actually done, that is, whether we shall be able to modify our concrete representation in such a manner that the work actually done will be equivalent to the fictitious work of our concrete representation.

As we know almost nothing about those regions where external electromotive forces reside (cf. p. 22), we shall

again be obliged to limit ourselves in the following to those regions where

$$X = Y = Z = 0.$$

For simplicity we shall moreover neglect ϵ in comparison to m (cf. p. 52). The quantity of positive fluid contained in the volume-element $d\tau$ will then be $\frac{md\tau}{2}$, and of negative $-\frac{md\tau}{2}$. According to our concrete representation each moves with the velocity u' along the x-axis, the former in the positive and the latter in the negative direction, and the force acting on each in the direction of its motion is $\frac{md\tau}{2}\xi$; the work done by these forces in overcoming friction and transformed into heat per volume-element $d\tau$ during the time dt will therefore be

$$md\tau(\xi u' + \eta v' + \zeta w')dt = BL(P^2 + Q^2 + R^2)d\tau\, dt.$$

The work actually done and transformed into heat (Joule's heat) is now given by formula (18, I.). In order that our concrete representation may therefore give the correct expression for the actual work done and transformed into Joule's heat we must put

$$B = 1. \qquad \qquad (10)$$

In those regions where external electromotive forces reside the amount of heat generated by the friction of the electric fluids would be

$$md\tau(\xi u' + \eta v' + \zeta w')dt$$
$$= L[(P+X)^2 + (Q+Y)^2 + (R+Z)^2]d\tau\, dt,$$

where the constant B has already been replaced by its value $B = 1$ determined from the special case $X = Y = Z = 0$. To reconcile this expression for the fictitious heat with that (14, II.) for Joule's heat we should then evidently have to assume the following expression for Λ:

$$\Lambda = L[(2P+X)X + (2Q+Y)Y + (2R+Z)Z]. \qquad (11)$$

In order to have such other values of Λ as might represent its real value it would be necessary to introduce new hypotheses into our concrete representation, for example, that the molecules of the body in those regions, where external electromotive forces reside, attract or repel the electric fluids according to some given law. As such special investigations are wanting in Maxwell's theory we shall feel justified in not pursuing them further here. Under the above assumptions and limitations our concrete representation will, however, give the correct expression for the energy transformed into heat. We have then

$$\xi = P + X, \quad \eta = Q + Y, \quad \zeta = R + Z. \qquad (12)$$

We designate the algebraic sum of all the fluids that pass during unit-time through unit-surface at right angles to the x, y, and z axes as the components p, q, and r respectively of the current-strength, and the algebraic sum of all the fluids that pass during unit-time through any unit-surface as its component ω, or if this surface is at right angles to the direction of flow, as the total current-strength Ω. By equations (7) we find then the following expressions for these quantities:

$$\left.\begin{aligned} p &= L(P+X) = L\xi, \quad q = L(Q+Y) = L\eta, \\ r &= L(R+Z) = L\zeta, \quad \omega = L(N+S), \\ \Omega &= \sqrt{p^2+q^2+r^2} \\ &= L\sqrt{(P+X)^2+(Q+Y)^2+(R+Z)^2} \\ &= L\sqrt{\xi^2+\eta^2+\zeta^2}; \\ \text{or} \quad \Omega &= L\nu_1 = L\sqrt{N_1{}^2+S_1{}^2+2N_1S_1\cos(N_1S_1)}; \end{aligned}\right\} \quad \ldots(13)$$

where ν_1, N_1, and S_1 are the vectors (ξ, η, ζ), (P, Q, R), and (X, Y, Z) respectively.

According to equation (2) the increment of the real electricity in any volume-element must be equal to the excess of the fluids flowing into it over those flowing out of it. This must now also hold for any finite region T

enclosed by any closed surface s. The real electricity contained in any such region T will be

$$\int \epsilon \, d\tau,$$

where the integration is to be extended throughout the region T. The quantity of fluid (positive and negative) entering it through the surface s during the time dt will be according to our concrete representation

$$dt \int \omega \, do,$$

where the integration is to be extended over the surface s, and the given normal to be drawn into the region T. We have therefore

$$\frac{d}{dt}\int \epsilon \, d\tau = \int \omega \, do,$$

or replacing ϵ and ω by their values (1) and (13) respectively,

$$\left.\begin{aligned} &\frac{1}{4\pi}\frac{d}{dt}\int d\tau\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right] \\ &\quad = \int do L[(P+X)\cos(n, x) \\ &\qquad +(Q+Y)\cos(n, y)+(R+Z)\cos(n, z)]. \end{aligned}\right\} \quad \ldots(14)$$

This equation can of course be derived directly from equation (2) without the aid of our concrete representation; for this purpose we replace ϵ in equation (2) by its value (1), multiply by $d\tau$ and integrate through the region T; we have then

$$\begin{aligned} &\frac{1}{4\pi}\frac{d}{dt}\int d\tau\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right] \\ &\quad = \int d\tau\left[\frac{d}{dx}L(P+X)+\frac{d}{dy}L(Q+Y)+\frac{d}{dz}L(R+Z)\right]; \end{aligned}$$

we next apply the following form of Green's law to the right-hand side of this equation, and find the desired equation directly:

$$\int d\tau\left[\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right]$$
$$=\int do[f\cos(n,x)+g\cos(n,y)+h\cos(n,z)].$$

Since the real and neutral electricities of any insulator, within which no electromotive forces reside, are, as we have seen above, bound, it follows that there can be no flow of electricity through it, and hence that in any system of bodies enclosed by such an insulator the total amount of real electricity can be neither increased nor diminished; equation (14) then assumes the special form

$$\frac{1}{4\pi}\frac{d}{dt}\int d\tau\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]=0,$$

which gives

$$\frac{1}{4\pi}\int d\tau\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]=\text{const.}$$

It follows, therefore, that the amount of positive fluid generated in any such system, for example, our universe, must always be exactly equal to the amount of negative fluid generated within it.

We have already remarked that the quantities P, Q, R, X, Y, Z can be measured in any other than the electrostatic system of units. Let us now denote quantities measured in any system of units by suffixing to them the index h and the factor of proportionality between the units of this new system and those of the electrostatic system by the constant h. We have then

$$\left.\begin{array}{lll} P=P_h/h, & Q=Q_h/h, & R=R_h/h, \\ X=X_h/h, & Y=Y_h/h, & Z=Z_h/h. \end{array}\right\} \quad \text{........(15)}$$

By the substitution of these values for P, Q, R, X, Y, Z in equations (7, II.) and (10, II.) the latter become

$$T=\frac{D}{8\pi h^2}(P_h^2+Q_h^2+R_h^2), \quad\ldots\ldots\ldots\ldots\ldots(16)$$

and

$$\left.\begin{aligned}\frac{hM}{\mathfrak{B}}\frac{d\alpha}{dt}=\frac{dR_h}{dy}-\frac{dQ_h}{dz}, \quad \frac{hM}{\mathfrak{B}}\frac{d\beta}{dt}=\frac{dP_h}{dz}-\frac{dR_h}{dx},\\ \frac{hM}{\mathfrak{B}}\frac{d\gamma}{dt}=\frac{dQ_h}{dx}-\frac{dP_h}{dy}.\end{aligned}\right\}\ldots(17)$$

It is now always desirable (cf. also § 16) to define the real electricity in any system in such a manner that P_h, Q_h, R_h are as above the forces that act on unit-quantity of electricity; for this purpose the density ϵ_h of the real electricity measured in this system must evidently be put equal to

$$\epsilon_h=\frac{\epsilon}{h}=\frac{1}{4\pi h^2}\left[\frac{d(DP_h)}{dx}+\frac{d(DQ_h)}{dy}+\frac{d(DR_h)}{dz}\right]\ldots.(18)$$

In our definitions of p_h, q_h, r_h, ω_h, and Ω_h it then becomes necessary to replace the electricity measured in the electrostatic system by that measured in the new system. We have then

$$p=hp_h,\ q=hq_h,\ r=hr_h,\ \omega=h\omega_h,\ \Omega=h\Omega_h;\ \ldots(19)$$

p_h, for example, is the quantity of electricity measured in the new system that passes through unit-surface at right angles to the x-axis during unit-time. In order to avoid a new constant in formulae (9, II.), (12, II.), (13, II.) and (13) we can introduce, instead of the so-called conductivity L measured in the electrostatic system, a new constant

$$L_h=L/h^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

the conductivity measured in the new system. We

find then

$$\left.\begin{aligned}\frac{D}{h\mathfrak{V}}\frac{dP_h}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}-4\pi\frac{hL_h}{\mathfrak{V}}(P_h+X_h),\\ \frac{D}{h\mathfrak{V}}\frac{dQ_h}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}-4\pi\frac{hL_h}{\mathfrak{V}}(Q_h+Y_h),\\ \frac{D}{h\mathfrak{V}}\frac{dR_h}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}-4\pi\frac{hL_h}{\mathfrak{V}}(R_h+Z_h),\end{aligned}\right\}\quad\ldots\ldots(21)$$

$$W=L_h(P_h^2+Q_h^2+R_h^2),\quad\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$\Gamma=-L_h(P_hX_h+Q_hY_h+R_hZ_h),\ldots\ldots\ldots(23)$$

$$\left.\begin{aligned}p_h=L_h(P_h+X_h),\ q_h=L_h(Q_h+Y_h),\ r_h=L_h(R_h+Z_h),\\ \omega=L_h(N_{1h}+S_{1h}),\\ \Omega_h=L_h\sqrt{N_{1h}^2+S_{1h}^2+2N_{1h}S_{1h}\cos(N_{1h},\,S_{1h})}.\end{aligned}\right\}\ldots(24)$$

CHAPTER IV.

SECTION IX. SIMILARITY OF THE ELECTRIC OSCILLATIONS TO THOSE OF LIGHT.

THE equations already developed represent in general a wave motion. For a homogeneous body, within which no external electromotive forces reside, that is, D and L constant, and $X = Y = Z = 0$, equations (9, II.) and (1, III.) give

$$\frac{d\epsilon}{dt} + \frac{4\pi L}{D}\epsilon = 0,$$

hence $$\epsilon = Ae^{-4\pi Lt/D}. \qquad\qquad (1)$$

If the body has been at any previous time in the unelectrified state, and no external electromotive forces have acted in the meantime within it, the relation (1) for ϵ must also have been valid during that period. As ϵ was then equal to zero, A must be equal to zero, and consequently $\epsilon = 0$ for all future periods. Since D is constant, it follows then from formula (1, III.) that

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = \theta = 0. \qquad\qquad (2)$$

Differentiating equations (9, II.), for example the first, with regard to t, and replacing $\frac{d\beta}{dt}$ and $\frac{d\gamma}{dt}$ by their values (10, II.), we find

$$\frac{D}{\mathfrak{V}}\frac{d^2P}{dt^2}=\frac{d}{dz}\frac{\mathfrak{V}}{M}\left(\frac{dP}{dz}-\frac{dR}{dx}\right)$$
$$-\frac{d}{dy}\frac{\mathfrak{V}}{M}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)-\frac{4\pi L}{\mathfrak{V}}\frac{d}{dt}(P+X),$$

or $$MD\frac{d^2P}{dt^2}+4\pi L\frac{dP}{dt}$$
$$=\mathfrak{V}^2\left[\left(\frac{d^2P}{dx^2}+\frac{d^2P}{dy^2}+\frac{d^2P}{dz^2}\right)-\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\right];$$

or, since here $\theta=0$,

and, similarly,
$$\left.\begin{aligned}MD\frac{d^2P}{dt^2}+4\pi ML\frac{dP}{dt}&=\mathfrak{V}^2\nabla^2P,\\ MD\frac{d^2Q}{dt^2}+4\pi ML\frac{dQ}{dt}&=\mathfrak{V}^2\nabla^2Q,\\ MD\frac{d^2R}{dt^2}+4\pi ML\frac{dR}{dt}&=\mathfrak{V}^2\nabla^2R.\end{aligned}\right\}\quad\text{........(3)}$$

For insulators the conductivity L is so small that the second terms of these equations (3) can be rejected in comparison to the other two terms, and we have

$$\left.\begin{aligned}MD\frac{d^2P}{dt^2}=\mathfrak{V}^2\nabla^2P,\quad MD\frac{d^2Q}{dt^2}=\mathfrak{V}^2\nabla^2Q,\\ MD\frac{d^2R}{dt^2}=\mathfrak{V}^2\nabla^2R.\end{aligned}\right\}\quad\text{.......(4)}$$

These equations of action for the ether are of exactly the same form as those for light; conversely, we can therefore conclude that the propagation of electric disturbances in media, for which the above conditions ($L=0$, etc.) hold, is effected by waves similar to those of light. In this similarity between the electric oscillations of the ether and those of light lies one confirmation of the assumption of the electro-magnetic nature of light-waves.

If L is very large, as in most metals, and D still comparatively small, the first terms of equations (3) can be rejected in comparison to the other two, and we have

$$\left.\begin{array}{c} 4\pi ML\dfrac{dP}{dt}=\mathfrak{V}^2\nabla^2 P, \quad 4\pi ML\dfrac{dQ}{dt}=\mathfrak{V}^2\nabla^2 Q, \\ 4\pi ML\dfrac{dR}{dt}=\mathfrak{V}^2\nabla^2 R. \end{array}\right\} \quad \ldots\ldots(5)$$

These equations are identical to the approximate equations for the transmission of heat through good conducting bodies, as copper. The transition from equations (4) to (5) corresponds to that from very small to very large values of L. L causes a damping of the electric waves, as we shall see directly.

To show more clearly the similarity between the electric waves and those of light, let us next examine particular integrals or solutions of the above equations (3); to these correspond, of course, only particular types or classes of waves out of the great multitude of those that are propagated through the ether.

Particular Solutions of Equations (4).—We suppose that the quantities P, Q, R of equations (4)—α, β, γ no longer appear as variables in these equations—are functions of x and t only, that is, that these variables have the same values in any given plane at right angles to the x-axis at any given time t; such a motion is a wave motion, the plane or surface, for which the variables have the same values, being designated as the wave proper. Since for this particular set of waves

$$\nabla^2=\frac{d^2}{dx^2}, \text{ and } \theta=\frac{dP}{dx}=0, \text{ hence } \frac{d^2P}{dx^2}=0,$$

$\dfrac{d}{dy}$ and $\dfrac{d}{dz}$ vanishing in conformity to the assumption that P, Q, R are functions of x and t only, our above equations (4) reduce to

$$\frac{d^2P}{dt^2}=0, \quad \frac{d^2Q}{dt^2}=\frac{\mathfrak{V}^2}{MD}\frac{d^2Q}{dx^2}, \quad \frac{d^2R}{dt^2}=\frac{\mathfrak{V}^2}{MD}\frac{d^2R}{dx^2}. \ldots\ldots(6)$$

The integral of the first of these equations is

$$P=At+B,$$

where A and B are arbitrary constants. Unless we put the constant $A=0$, this solution would denote a constant acceleration of the component F of the tone $(F, G, H,)$ that is, an infinite medium-energy. The only possible form of this solution could therefore be

$$P=B,$$

which would correspond to a cónstant velocity of the component F of the tone (F, G, H); we could explain such a stationary state of the ether by conceiving that our solar system were placed between the layers of an enormous condenser. According to the first interpretation of our fundamental expressions (cf. § 1) such a solution would denote that the ether of our solar system were advancing with a constant velocity through space, whereas according to our second interpretation it would correspond to a constant angular velocity (rotation) of the rotating granules of § 1; neither case can however concern us here, since we are really only considering its periodic or oscillatory motions.

To find the integrals of the last two equations (6) we make use of the expression

$$d\left(\frac{dQ}{dt}+\lambda\frac{dQ}{dx}\right), \qquad \text{.......................(7)}$$

where λ is an arbitrary constant. Performing the indicated differentiation we get

$$d\left(\frac{dQ}{dt}+\lambda\frac{dQ}{dx}\right)=dt\left(\frac{d^2Q}{dt^2}+\lambda\frac{d^2Q}{dt\,dx}\right)+dx\left(\frac{d^2Q}{dx\,dt}+\lambda\frac{d^2Q}{dx^2}\right).$$

Replacing here $\frac{d^2Q}{dx^2}$ by its value from equations (6), and assuming now for λ the value $\lambda=\frac{\mathfrak{B}^2}{MD}$, we have

$$d\left(\frac{dQ}{dt}+\lambda\frac{dQ}{dx}\right)=\left(\frac{1}{\lambda}\frac{d^2Q}{dt^2}+\frac{d^2Q}{dx\,dt}\right)\left(dx+\lambda dt\right), \qquad \text{......(8)}$$

which is only another form of the second of our differential equations (4). Here λ has two values

$$\lambda_1 = \frac{\mathfrak{B}}{\sqrt{MD}} = a, \quad \lambda_2 = -\frac{\mathfrak{B}}{\sqrt{MD}} = -a.$$

By introducing the new variable

$$u = x + \lambda t,$$

we can write (7) in the form

$$d\left(\frac{dQ}{dt} + \lambda\frac{dQ}{dx}\right) = \left(\frac{1}{\lambda}\frac{d^2Q}{dt^2} + \frac{d^2Q}{dx\,dt}\right)du,$$

from which it follows that

$$\frac{dQ}{dt} + \lambda\frac{dQ}{dx} = \Phi(u).$$

As this equation must be valid for both values of λ, a, and $-a$, we get the two equations

$$\frac{dQ}{dt} + a\frac{dQ}{dx} = \Phi(u) = \Phi(x + at),$$

$$\frac{dQ}{dt} - a\frac{dQ}{dx} = \Psi(u) = \Psi(x - at);$$

these give $$\frac{dQ}{dt} = \tfrac{1}{2}\Phi(x + at) + \tfrac{1}{2}\Psi(x - at),$$

and $$\frac{dQ}{dx} = \frac{1}{2a}\Phi(x + at) - \frac{1}{2a}\Psi(x - at). \qquad \text{............(9)}$$

The integrals of these equations are

$$Q = \tfrac{1}{2}\int\Phi(x + at)dt + \tfrac{1}{2}\int\Psi(x - at)dt + \mathrm{f}(x)$$

and $$Q = \tfrac{1}{2}a\int\Phi(x + at)dx - \tfrac{1}{2}a\int\Psi(x - at)dx + \mathrm{g}(t),$$

where f(x) and g(t) are arbitrary functions of x and t respectively, which can always be made to vanish by proper choice of the initial values of x and of t; we can thus write the last two equations

$$\left.\begin{aligned} Q&=\tfrac{1}{2}\int\Phi(x+at)dt+\tfrac{1}{2}\int\Psi(x-at)dt \\ \text{and}\qquad Q&=\tfrac{1}{2}a\int\Phi(x+at)dx-\tfrac{1}{2}a\int\Psi(x-at)dx \end{aligned}\right\},\ldots\ldots(10)$$

or $$Q=\phi(x+at)+\psi(x-at), \ldots\ldots\ldots\ldots(11)$$

and similarly

$$R=\chi(x+at)+\Omega(x-at), \ldots\ldots\ldots\ldots(12)$$

where ϕ and χ and ψ and Ω are arbitrary functions of $(x+at)$ and $(x-at)$ respectively. Not only these integrals (11) and (12) but also their separate terms ϕ, ψ, χ, Ω are particular solutions of the above equations (6); this follows directly from their substitution in the latter.

The interpretation of such particular integrals, for example

$$Q=\psi(x-at) \quad\text{or}\quad R=\Omega(x-at),$$

is extremely simple; if, namely, t increases by τ and x by $a\tau$, ψ remains unaltered

$$\psi[x+a\tau-a(t+\tau)]=\psi(x-at),$$

as also R. We see, therefore, that the state of the ether at the time t at the distance x from the source of the disturbance is exactly the same as that which will prevail at the time $(t+\tau)$ at the distance $(x+a\tau)$, and that the ether at the distance $(x+a\tau)$ will not have reached this state until this period $t+\tau$ has elapsed. The disturbance or wave is therefore propagated along the x-axis with the velocity a. Similarly, $R=\Omega(x-at)$

represents a wave advancing along the x-axis with exactly the same velocity a as Q but entirely independent of it. We see, therefore, that the motion characterized by Q and R is in every respect similar to the wave motion of light. Similarly, the particular integrals $Q=\phi(x+at)$, $R=\chi(x+at)$ represent waves advancing with the same velocity a as $Q=\Psi(x-at)$, $R=\Omega(x-at)$, but in exactly the opposite direction. The interpretation of the general integrals (11) and (12) of the particular equations (6) is evident from the above.

Let us next examine the particular solution of equations (3), where P, Q, R are functions of x and t only. As these equations are linear, their particular integrals will be exponential functions and for the case in question of the form

$$Ae^{(p+iq)x+int}, \qquad (13)$$

where p, q, n are arbitrary (real) constants to be determined later. The introduction of the imaginary unit i in the exponent of e is necessary in order that the function be periodic.

For the case in question our differential equations (3) assume the special form

$$\left.\begin{aligned} MD\frac{d^2P}{dt^2}+4\pi ML\frac{dP}{dt}&=0,\\ MD\frac{d^2Q}{dt^2}+4\pi ML\frac{dQ}{dt}&=\mathfrak{B}^2\frac{d^2Q}{dx^2},\\ MD\frac{d^2R}{dt^2}+4\pi ML\frac{dR}{dt}&=\mathfrak{B}^2\frac{d^2R}{dx^2}.\end{aligned}\right\} \qquad (14)$$

Assuming the above general solution (13) for Q, and substituting it in these differential equations we find the following conditional equation between the arbitrary exponential constants:

$$-MDn^2+4\pi MLni=\mathfrak{B}^2(p+iq)^2, \qquad (15)$$

or since its real and imaginary parts must be entirely independent of each other, the two relations

$$-MDn^2=\mathfrak{V}^2(p^2-q^2) \text{ and } 4\pi MLni=2\mathfrak{V}^2pqi, \ldots(16)$$

Eliminating p from these two equations we have

$$\frac{4\pi^2n^2M^2L^2}{\mathfrak{V}^2q^2}-q^2+\frac{MDn^2}{\mathfrak{V}^2}=0$$

or
$$(q/n)^4-\frac{MD}{\mathfrak{V}^2}(q/n)^2-\frac{4\pi^2M^2L^2}{n^2\mathfrak{V}^2}=0,$$

from which it follows that

$$(q/n)^2=\frac{MD}{2\mathfrak{V}^2}+\sqrt{\frac{M^2D^2}{4\mathfrak{V}^4}+\frac{4\pi^2M^2L^2}{n^2\mathfrak{V}^2}}=\frac{1}{a^2} \ldots\ldots\ldots(17)$$

The plus sign has been chosen in extracting this square root, as otherwise $(q/n)^2$ would become negative, and hence (q/n) imaginary, which is not in conformity to our above assumption that q and n are real numbers (constants).

The expression

$$Q=Ae^{(p+iq)x+int}$$

—where $q=\frac{n}{a}$ is determined by equation (17) and p by equation (16); n is still entirely arbitrary, it determines the wave-length of the waves imparted, as we shall see directly—is therefore a particular integral or solution of the above differential equation (14) for Q.

We can now write Q in the form

$$Q=Ae^{px}e^{(qx+nt)i}=Ae^{px}[\cos(qx+nt)+i\sin(qx+nt)],$$

and not only the real part of this expression, namely,

$$Ae^{px}\cos(qx+nt),$$

but also the imaginary factor

$$Ae^{px}\sin(qx+nt)$$

must satisfy our differential equation for Q.

Let us examine the former solution more carefully. We replace p and n by their values (17), and have

$$n=-aq,\quad p=-\frac{2\pi ML}{\mathfrak{V}^2}a \quad\text{.................(18)}$$

—we choose here the negative root of the equation $(q/n)^2=\frac{1}{a^2}$ for (q/n); we find then

$$Q=Ae^{-\frac{2\pi ML}{\mathfrak{V}^2}ax}\cos q(x-at). \quad\text{...............(19)}$$

Now let t increase by τ and x by $a\tau$; we have then

$$Q(t+\tau,\, x+a\tau)=Ae^{-\frac{2\pi ML}{\mathfrak{V}^2}ax}e^{-\frac{2\pi ML}{\mathfrak{V}^2}a^2\tau}\cos q(x-at),$$

hence $$Q(t+\tau,\, x+a\tau)=e^{-\frac{2\pi ML}{\mathfrak{V}^2}a^2\tau}Q(t,x). \quad\text{..........(20)}$$

The cosine remains unaltered, but the exponential factor diminishes as x increases. The phases that appear at any point of the x-axis will therefore reappear after an elapse of the time τ at the distance $a\tau$ further along it, these phases diminishing however in magnitude as x increases. Their rate of diminution or damping, though due directly to the value of the exponential factor, depends indirectly on the value of L. Where L is 0 or very small, this factor is almost equal to unity for all values of x, decreasing only inappreciably as x becomes very large; very small values of L will therefore cause hardly any damping, and the waves will advance quite unimpeded with each amplitude equal or almost equal to the preceding one. As L increases, the successive amplitudes decrease more and more rapidly as x increases, until finally for very large values of L the damping becomes so considerable that the wave motion is extinguished almost immediately. This damping of

the electric waves corresponds precisely to the absorption of the luminiferous waves.

For a given value of L the velocity of propagation a of the waves remains constant (cf. formula (17)). Even for $L=0$ this velocity a is enormously large (that of light). As L increases, a increases; consequently $\sqrt{DM}/\mathfrak{V}$ is as small as or smaller than the reciprocal value of the velocity of propagation of these waves. From formula (19) we see that the same phase returns when x increases by $\frac{2\pi}{q}$; $\frac{2\pi}{q}$ is therefore the wave length.

Besides the difference just mentioned between those waves where $L=0$ and those where $L \gtrless 0$, namely, a damping (absorption) in the latter case and none in the former, another very important distinction should be observed. For $L=0$, $a=\frac{\mathfrak{V}}{\sqrt{DM}}$, that is, the velocity of propagation of the waves depends only on the constitution of the medium, its constants D and M, and not for example on the wave length. This is, however, no longer the case as soon as $L \gtrless 0$, for a then becomes a function not only of the medium but also of n the arbitrary constant (cf. formula (17)); as n increases the wave length $\lambda=\frac{2\pi}{q}=-\frac{2\pi a}{n}$ (cf. formula (18)) decreases and the velocity a increases. We see therefore that for $L \gtrless 0$ waves of different wave length advance with different velocities, those of greater wave length with smaller velocity and conversely. Any given motion that is imparted to the medium can now be expressed by a Fourier's series, a given wave length being represented by every term of this series. When an arbitrary motion is imparted to a medium for which $L \gtrless 0$, the different terms of the Fourier's series representing this motion will therefore advance with different velocities, their wave lengths being different; in advancing these waves will consequently partly destroy one another by inter-

ference and partly be superposed; the curve representing the ensuing motion will therefore be correspondingly irregular. This interference or dispersion of the electric waves corresponds precisely to the dispersion of light waves. We should observe, however, that this phenomenon of dispersion is to be found only in media for which $L \gtreqless 0$.

SECTION X. THEORY OF THE HERTZIAN OSCILLATIONS.

In examining the electrodynamic state of the ether during the passage of electric sparks between the brass-balls of a Hertzian vibrator, we shall take the ideal case: we suppose the vibrator to be a body of rotation and the disturbance to be propagated radially with regard to it; if we choose its axis of rotation as z-axis and the centre of the spark-gap as origin of our system of coordinates,

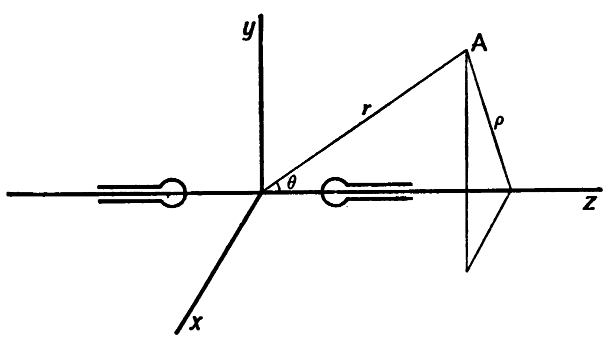

FIG. 5.

as indicated in the annexed figure, everything will be symmetrical with regard to this axis; the resultant electric force at any point A will thus act in a plane passing through that point and the given axis. We resolve this force into two components, one parallel to

the z-axis and the other at right angles to it; we denote the latter by S and have

$$P=S\cos(\rho,\,x)=S\frac{x}{\rho},\quad Q=S\cos(\rho,\,y)=S\frac{y}{\rho},$$

where $\rho=\sqrt{x^2+y^2}$ denotes the distance of the given point from the z-axis. As S is evidently a function of only ρ, z, and t, we can write

$$P=\frac{x}{\rho}S(\rho,\,z,\,t),\quad Q=\frac{y}{\rho}S(\rho,\,z,\,t),$$

or if we put

$$\int S(\rho,\,z,\,t)d\rho=s(\rho,\,z,\,t),$$

hence

$$\frac{ds}{d\rho}=S(\rho,\,z,\,t),$$

$$P=\frac{ds}{dx},\quad Q=\frac{ds}{dy}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

The electrodynamic state of the surrounding ether is now defined by Maxwell's fundamental equations (9, II.) and (10, II.), which assume here the following special form:

$$\left.\begin{aligned}\frac{1}{\mathfrak{V}}\frac{dP}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}\\ \frac{1}{\mathfrak{V}}\frac{dQ}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\\ \frac{1}{\mathfrak{V}}\frac{dR}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}\end{aligned}\right\},\ldots(22)\qquad \left.\begin{aligned}\frac{1}{\mathfrak{V}}\frac{d\alpha}{dt}&=\frac{dR}{dy}-\frac{dQ}{dz}\\ \frac{1}{\mathfrak{V}}\frac{d\beta}{dt}&=\frac{dP}{dz}-\frac{dR}{dx}\\ \frac{1}{\mathfrak{V}}\frac{d\gamma}{dt}&=\frac{dQ}{dx}-\frac{dP}{dy}\end{aligned}\right\}\ldots(23)$$

For the given form of vibrator the last of these equations evidently reduces to

$$\frac{d\gamma}{dt}=0,$$

hence

$$\gamma=\text{const.},$$

or, if we assume that γ was initially zero,

$$\gamma = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

The differentiation of equations (23), the first with regard to x, the second to y, and the third to z, and their addition give the following relation:

$$\frac{1}{\mathfrak{V}}\frac{d}{dt}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0,$$

hence $$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=\text{const.}=0.$$

By formula (24) this relation reduces to

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}=0,$$

which gives $$-\beta\, dx+\alpha\, dy=d\chi(x, y, z, t),$$

a complete differential, and hence

$$\alpha=\frac{d\chi}{dy}, \quad \beta=-\frac{d\chi}{dx}, \quad \gamma=0. \quad \ldots\ldots\ldots\ldots(25)$$

By the introduction of this function equations (22) can be written

$$\frac{1}{\mathfrak{V}}\frac{dP}{dt}=-\frac{d^2\chi}{dx\,dz}, \quad \frac{1}{\mathfrak{V}}\frac{dQ}{dt}=-\frac{d^2\chi}{dy\,dz}, \quad \frac{1}{\mathfrak{V}}\frac{dR}{dt}=\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2},$$

or integrated

$$\left.\begin{aligned} P=-\mathfrak{V}\frac{d^2}{dx\,dz}\int\chi dt, \quad Q=-\mathfrak{V}\frac{d^2}{dy\,dz}\int\chi dt, \\ R=\mathfrak{V}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\int\chi dt \end{aligned}\right\} \ldots(26)$$

Comparing these values for P, Q, R with those (21) above we find

$$\frac{ds}{dx}+\mathfrak{V}\frac{d^2}{dx\,dz}\int\chi dt=0 \quad \text{and} \quad \frac{ds}{dy}+\mathfrak{V}\frac{d^2}{dy\,dz}\int\chi dt=0,$$

which give

$$s+\mathfrak{V}\frac{d}{dz}\int\chi\,dt=\mathrm{f}(z,t);$$

that is, the expression on the left is a function of z and t only.

This last relation can now be written

$$\mathfrak{V}\int\chi\,dt=-\int s\,dz+\int\mathrm{f}(z,t)\,dz=\Pi(\rho,z,t)+\eta(z,t),$$

hence $$\chi=\frac{1}{\mathfrak{V}}\frac{d}{dt}[\Pi(\rho,z,t)+\eta(z,t)]. \qquad\qquad (27)$$

We next substitute this value for χ in formulae (25) and (26) and find

$$a=\frac{1}{\mathfrak{V}}\frac{d^2\Pi}{dy\,dt},\quad \beta=-\frac{1}{\mathfrak{V}}\frac{d^2\Pi}{dx\,dt},\quad \gamma=0, \qquad (28)$$

and

$$P=-\frac{d^2\Pi}{dx\,dz},\quad Q=-\frac{d^2\Pi}{dy\,dz},\quad R=\left(\frac{d^2\Pi}{dx^2}+\frac{d^2\Pi}{dy^2}\right); \qquad (29)$$

we observe that the function η no longer appears in the given expressions. It thus follows that the six variables P, Q, R, a, β, γ which determine the electromagnetic state of the field can be expressed in terms (derivatives) of a single function $\Pi(\rho, z, t)$.

To determine the function Π we substitute the values (28) and (29) for P, Q, R, a, β, γ in the first two of our fundamental equations (23), and have

$$\frac{d}{dy}\left[\frac{d^2\Pi}{dt^2}-\mathfrak{V}^2\nabla^2\Pi\right]=\frac{d}{dx}\left[\frac{d^2\Pi}{dt^2}-\mathfrak{V}^2\nabla^2\Pi\right]=0,$$

hence $$\frac{d^2\Pi}{dt^2}-\mathfrak{V}^2\nabla^2\Pi=\mathrm{f}(z,t),$$

as differential equation for the determination of our function Π. We have now just found that the values of the given variables are in no way effected by the

appearance of an additive function $\eta(z, t)$ to Π; the following differential equation will therefore evidently suffice for the determination of this function,

$$\frac{d^2\Pi}{dt^2} - \mathfrak{V}^2\nabla^2\Pi = 0. \qquad\qquad (30)$$

Before examining particular integrals of this differential equation, let us determine the equation for the lines of electric force in the given field corresponding to its general integral. As everything is symmetrical with regard to the z-axis, it will evidently suffice, if we determine the equation for the lines of force in any meridian-plane passing through the z-axis, for example, the xz coordinate-plane. The general expression for the lines of electric force is now

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R};$$

for the given plane, $y=0$, $x=\rho$, this relation reduces to

$$\frac{d\rho}{P} = \frac{dz}{R}.$$

Replace here P and R by their respective values (29) and we get

$$\frac{d^2\Pi}{d\rho\, dz}dz + \left[\frac{d^2\Pi}{d\rho^2} + \left(\frac{d^2\Pi}{dy^2}\right)_{y=0}\right]d\rho = 0,$$

or, since
$$\frac{d\Pi}{dy} = \frac{d\Pi}{d\rho}\frac{d\rho}{dy} = \frac{d\Pi}{d\rho}\, y/\rho,$$

and hence
$$\left(\frac{d^2\Pi}{dy^2}\right)_{y=0} = \frac{1}{\rho}\frac{d\Pi}{d\rho},$$

$$\frac{d^2\Pi}{d\rho\, dz}dz + \left(\frac{d^2\Pi}{d\rho^2} + \frac{1}{\rho}\frac{d\Pi}{d\rho}\right)d\rho = 0.$$

This expression becomes a complete differential if we multiply by ρ; we have then

$$\rho\frac{d^2\Pi}{d\rho\, dz}dz + \left(\rho\frac{d^2\Pi}{d\rho^2} + \frac{d\Pi}{d\rho}\right)d\rho = d\left(\rho\frac{d\Pi}{d\rho}\right) = 0,$$

hence $$\rho\frac{d\Pi}{d\rho}=\text{const.}, \qquad (31)$$

which is the desired equation for the lines of electric force in the $xz=\rho z$ plane.

The differential equation (30) includes all disturbances propagated through the surrounding dielectric (air) by the given vibrator, these being represented by its various particular integrals. Only those solutions are of interest to us here that exhibit the characteristics of the Hertzian oscillations. To obtain such we assume that Π, which is in general a function of ρ, z, and t, is a function of r only, the distance of any point from the origin; this condition evidently excludes all disturbances of a non-radial direction of propagation from our system. We have then

$$\nabla^2\Pi=\frac{d^2\Pi}{dr^2}+\frac{2}{r}\frac{d\Pi}{dr},$$

and our differential equation (30) assumes the special form

$$\frac{d^2\Pi}{dt^2}-\mathfrak{V}^2\left(\frac{d^2\Pi}{dr^2}+\frac{2}{r}\frac{d\Pi}{dr}\right)=0;$$

or, if we put $\Pi=\frac{u}{r}$, the familiar form

$$\frac{d^2u}{dt^2}-\mathfrak{V}^2\frac{d^2u}{dr^2}=0, \qquad (32)$$

the general integral of this equation is

$$u=f(r-\mathfrak{V}t)+g(r+\mathfrak{V}t),$$

hence $$\Pi=\frac{f(r-\mathfrak{V}t)}{r}+\frac{g(r+\mathfrak{V}t)}{r},$$

where f and g are arbitrary functions of the parameters $(r-\mathfrak{V}t)$ and $(r+\mathfrak{V}t)$ respectively. We next seek those

particular solutions of Π, that is, those forms of the function f and g, that possess the desired properties. One such solution is

$$f = El \sin m(r - \mathfrak{V}t), \quad \Pi = \frac{El \sin m(r - \mathfrak{V}t)}{r}, \quad \ldots(34)$$

where El is a constant and m an inverse length, $m = \frac{\pi}{\lambda}$.

We shall see in the following article that when any given electric disturbance or more precisely the rate at which the successive impulses giving rise to this disturbance are imparted to any given medium, is slow in comparison to its velocity of propagation, no waves will appear in the system; we shall then characterize the given state of the ether as stationary (aphotic)—compare also the analogous case for the vibrating cord in § 11. We shall, however, find that this is not the only condition for aphotic motion, for take the elastic cord of § 11 and impart to its free end a very rapid vibratory motion, the cord will still remain uniformly stretched, provided it is very short, that is, provided its length and the amplitude of the impulses imparted to it are of the same dimensions. The analogous holds now for the electric disturbances emitted by the Hertzian vibrator, namely, that the ether in the immediate neighbourhood of the given vibrator remains under uniform stress; this follows from the special form assumed by our equations of action in that region (cf. also text, p. 87); we write Π in the form

$$\Pi = \frac{El \sin m\left(r - \frac{\lambda}{\tau}t\right)}{r},$$

where $$\tau = \frac{\lambda}{\mathfrak{V}}.$$

In the immediate neighbourhood of the vibrator r will evidently be small in comparison to λ, and hence mr

small in comparison to $m\vartheta t$, so that we can reject the term mr in $\sin m(r - \frac{\lambda}{\tau}t)$ and thus write

$$\Pi = -\frac{El \sin \frac{m\lambda}{\tau}t}{r} \quad \text{.....................(35)}$$

Formulae (29) then give the following values for P, Q, R in the region in question;

$$P = El \sin \frac{m\lambda}{\tau}t \frac{d^2}{dx\,dz}\left(\frac{1}{r}\right), \quad Q = El \sin \frac{m\lambda}{\tau}t \frac{d^2}{dy\,dz}\left(\frac{1}{r}\right),$$

$$R = -El \sin \frac{m\lambda}{\tau}t\left[\frac{d^2}{dx^2}\left(\frac{1}{r}\right) + \frac{d^2}{dy^2}\left(\frac{1}{r}\right)\right],$$

or, since $\nabla^2\left(\frac{1}{r}\right) = 0$,

$$R = El \sin \frac{m\lambda}{\tau}t \frac{d}{dz^2}\left(\frac{1}{r}\right),$$

from which expressions it follows that

$$P = -\frac{d\phi}{dx}, \quad Q = -\frac{d\phi}{dy}, \quad R = -\frac{d\phi}{dz},$$

where $$\phi = -El \sin \frac{m\lambda}{\tau}t \frac{d}{dz}\left(\frac{1}{r}\right), \quad \text{..................(36)}$$

that is, that P, Q, R have a potential ϕ in the given region, and hence that all electric oscillations must be excluded from it.

To interpret the expression (36) for ϕ, we first determine the potential V at any point of the given region due to the presence of two quantities of electricity E and $-E$ at the infinitely short distance $l/2$ on either side of the origin on the z-axis. If we denote the distances of the given quantities of electricity from the point, at which the potential V is sought, by

$$r_1 = \sqrt{x^2 + y^2 + (z - l/2)^2} = \sqrt{r^2 - lz}$$

and $$r_2 = \sqrt{x^2 + y^2 + (z + l/2)^2} = \sqrt{r^2 + lz},$$

we have
$$V=E\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

We next develop this expression according to ascending powers of l and we find

$$V=E\left[\left(\frac{1}{r}+\frac{lz}{2r^3}\right)-\left(\frac{1}{r}-\frac{lz}{2r^3}\right)\right]=-El\frac{d}{dz}\left(\frac{1}{r}\right).$$

From a comparison of these expressions for ϕ and V it is evident that the former corresponds to a so-called electric double-point, whose axis is the z-axis and whose moment oscillates with the period τ between the values El and $-El$; the distribution of the electric forces in the given region can thus be conceived as produced by a rectilinear oscillation of infinitely small amplitude l and period of oscillation τ, at whose poles the quantities of electricity $\pm E\sin nt$ are liberated.

The resultant magnetic force at any point of the given region—this acts at right angles to the z-axis—is by formula (28)

$$\sqrt{\alpha^2+\beta^2}=-\frac{1}{\mathfrak{A}}\sqrt{\left(\frac{d^2\Pi}{dy\,dt}\right)^2+\left(\frac{d^2\Pi}{dx\,dt}\right)^2}.$$

Replace here Π by its value (35), and we get

$$\sqrt{\alpha^2+\beta^2}=-Elm\cos\frac{m\lambda}{\tau}t\sqrt{\left[\frac{d}{dx}\left(\frac{1}{r}\right)\right]^2+\left[\frac{d}{dy}\left(\frac{1}{r}\right)\right]^2}$$
$$=-Elm\cos\frac{m\lambda}{\tau}t\,\rho/r^3;$$

or, if we use the polar coordinates

$$\rho=r\sin\theta,\quad z=r\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

(cf. figure 5, p. 74),

$$\sqrt{\alpha^2+\beta^2}=\frac{-Elm\cos\frac{m\lambda}{\tau}t\sin\theta}{r^2}.\ \ldots\ldots\ldots\ldots\ldots(38)$$

According to Biot-Savart's law (cf. formulae (36 and 37, XI.)) this is evidently the magnetic force arising from a current-element of length l along the z-axis, whose current-strength oscillates between the values

$$\pm\frac{\pi E}{\lambda}=\pm\frac{\pi E}{\tau\mathfrak{V}}.$$

Besides the regions just considered there are certain other regions of the given field, in which our formulae assume simple forms easily accessible to examination. We first form the following expressions, derivatives of Π, which we shall constantly need in the ensuing special cases:

$$\left.\begin{aligned}
\frac{d\Pi}{dx}&=a\frac{d}{dx}\left(\frac{1}{r}\right)+bm\frac{x}{r^2}=-\frac{x}{r^2}\left(\frac{a}{r}-bm\right),\\
\frac{d\Pi}{dy}&=-\frac{y}{r^2}\left(\frac{a}{r}-bm\right),\\
\frac{d^2\Pi}{dx\,dz}&=a\frac{d^2}{dx\,dz}\left(\frac{1}{r}\right)+bm\frac{z}{r}\frac{d}{dx}\left(\frac{1}{r}\right)+bm\frac{d}{dz}\left(\frac{x}{r^2}\right)-am^2\frac{xz}{r^3}\\
&=\frac{3xz}{r^5}a-\frac{3xz}{r^4}bm-\frac{xz}{r^3}am^2,\\
\frac{d^2\Pi}{dy\,dz}&=\frac{3yz}{r^5}a-\frac{3yz}{r^4}bm-\frac{yz}{r^3}am^2,\\
\frac{d^2\Pi}{dx^2}&=a\frac{d^2}{dx^2}\left(\frac{1}{r}\right)+bm\frac{x}{r}\frac{d}{dx}\left(\frac{1}{r}\right)+bm\frac{d}{dx}\left(\frac{x}{r^2}\right)-am^2\frac{x^2}{r^3}\\
&=-\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)a+\left(\frac{1}{r^2}-\frac{3x^2}{r^4}\right)bm-\frac{x^2}{r^3}am^2,\\
\frac{d^2\Pi}{dy^2}&=-\left(\frac{1}{r^3}-\frac{3y^2}{r^5}\right)a+\left(\frac{1}{r^2}-\frac{3y^2}{r^4}\right)bm-\frac{y^2}{r^3}am^2,\\
\frac{d^2\Pi}{dx\,dt}&=\frac{m\mathfrak{V}x}{r^2}\left(\frac{b}{r}+am\right)\quad\text{and}\quad\frac{d^2\Pi}{dy\,dt}=\frac{m\mathfrak{V}y}{r^2}\left(\frac{b}{r}+am\right),
\end{aligned}\right\}\tag{39}$$

where

$$a=El\sin m(r-\mathfrak{V}t)\quad\text{and}\quad b=El\cos m(r-\mathfrak{V}t).$$

We examine then the following special regions:

(1) The z-axis. We put $x=y=0$ in formulae (39), substitute these values for the respective derivatives in formulae (28) and (29), and we find

$$\alpha=\beta=\gamma=P=Q=0, \quad R=\frac{2}{r^2}\left(bm-\frac{a}{r}\right).$$

It follows therefore that the resultant electric force acts along the z-axis, diminishing inversely as the third power of the distance ($r=z$) for small values of z and inversely as its square at greater distances.

(2) The equatorial or xy coordinate-plane. We replace the respective derivatives of formulae (28) and (29) by their values (39), put $z=0$ and we find

$$\sqrt{\alpha^2+\beta^2}=\frac{m}{r}\left(\frac{b}{r}+am\right), \quad \gamma=0,$$

$$P=Q=0, \quad R=\frac{1}{r}\left(\frac{a}{r^2}-\frac{bm}{r}-am^2\right);$$

the resultant magnetic force in the equatorial-plane thus acts along the vector ρ, at right angles to the resultant electric force, whose direction of action is parallel to the z-axis. As we recede from the vibrator in the equatorial-plane, the electric force evidently diminishes at first rapidly, inversely as the third power of the distance, and finally slowly, inversely as its first power. From a comparison with the preceding case it follows therefore that at great distances from the vibrator the action of the electric force will only be perceptible in the equatorial-plane.

(3) Regions at great distances from the origin. The higher powers of $\left(\frac{1}{r}\right)$ in formulae (39) can then be rejected in comparison to its lower ones; retaining only the lowest power of $\left(\frac{1}{r}\right)$ we find by formulae (28) and (29)

$$\sqrt{\alpha^2+\beta^2}=\frac{am\sin\theta}{r},\quad \gamma=0,$$

$$\sqrt{P^2+Q^2}=\frac{am^2\sin\theta\cos\theta}{r},\quad R=-\frac{am^2\sin^2\theta}{r}.$$

These values give the equation

$$R\cos\theta+\sqrt{P^2+Q^2}\sin\theta=0,$$

that is, the direction of the electric force in the given regions is always at right angles to the vector r; the propagation of the given disturbance at great distances from the vibrator thus takes place in the form of transverse waves (cf. also text p. 88). The magnitude of the resultant electric force at any distant point is evidently

$$\frac{am^2\sin\theta}{r}=\frac{am^2\rho}{r^2},$$

hence it follows that the resultant electric force in any point at the constant distance r from the vibrator diminishes in magnitude directly as the distance ρ of that point from the z-axis.

To examine the remaining regions of the given field we employ equation (31) for the lines of electric force in the xz coordinate-plane. For the given function $\Pi=El\sin m(r-\mathfrak{V}t)$ we have

$$\begin{aligned}\frac{d\Pi}{d\rho}&=El\frac{d}{dr}\left[\frac{\sin m(r-\mathfrak{V}t)}{r}\right]\frac{dr}{d\rho}\\&=El\left[\frac{m\cos m(r-\mathfrak{V}t)}{r}-\frac{\sin m(r-\mathfrak{V}t)}{r^2}\right]\rho/r,\end{aligned}$$

which gives

$$\rho\frac{d\Pi}{d\rho}=El\left[\frac{m\cos m(r-\mathfrak{V}t)}{r}-\frac{\sin m(r-\mathfrak{V}t)}{r^2}\right]\rho^2/r=\text{const.},$$

or in the polar coordinates r and θ

$$\left[m\cos m(r-\mathfrak{V}t)-\frac{1}{r}\sin m(r-\mathfrak{V}t)\right]\sin^2\theta=\text{const.}\quad\ldots(40)$$

The actual construction of this complicated system of

curves has been made by Hertz* for the periods $t=0$, $\frac{\tau}{4}, \frac{\tau}{2}, \frac{3\tau}{4}$; we insert his four figures here:

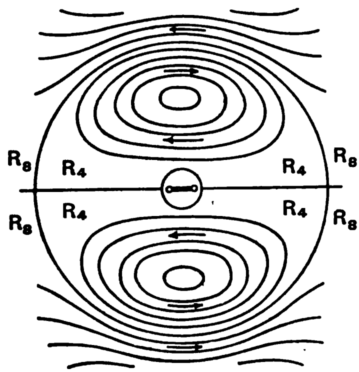

FIG. 6.

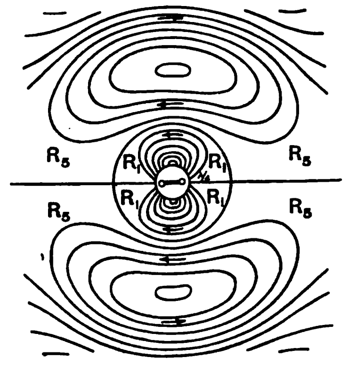

FIG. 7.

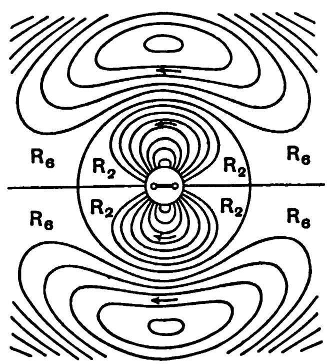

FIG. 8.

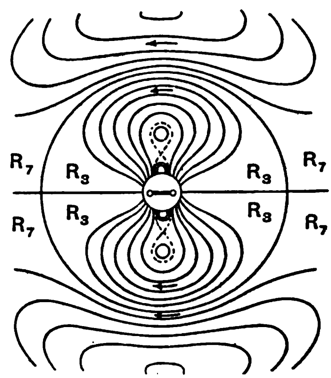

FIG. 9.

The arrows give the direction of action of the given electric forces; by suitable reversal of their directions these constructions will also hold for all whole multiple

* Wiedemann's *Annalen*, v. 36, 1889, plate 1.

periods of $\frac{\tau}{4}$. For $t=0$, figure 6, the current attains its maximum current-strength—this follows from formula (38)—whereas the poles of the rectilinear oscillation have then no electric charge whatever, that is, no lines of force either start from or end in them (cf. formula (36)). From the time $t=0$ lines of force begin to shoot out of the poles; they are enclosed in a sphere given by the equation $\rho\frac{d\Pi}{d\rho}=\text{const.}=0$; for $t=0$ this sphere is infinitely small, but it increases rapidly during the period $t=0-\frac{\tau}{4}$, at which latter value it occupies the region indicated by the R_1's in figure 7. We observe that the distribution of the lines of force within the given sphere is then approximately that of a corresponding electrostatic charge of the given poles. The velocity with which the sphere $\rho\frac{d\Pi}{d\rho}=0$ expands during the period $t=0-\frac{\tau}{4}$ is considerably greater than the velocity of light $\mathfrak{V}$, the latter is in fact of such dimensions that it would only reach the point marked $\frac{\lambda}{4}$ in figure 7 during this period. From $t=\frac{\tau}{4}$ the velocity of expansion of the given sphere continues to diminish in magnitude until it finally declines to the velocity $\mathfrak{V}$. At $t=\frac{\tau}{2}$ this sphere occupies the region R_2 of figure 8. Not only the electrostatic charge on the two poles but also the number of lines of force extending between them is then at a maximum. After the period $t=\frac{\tau}{2}$ the number of lines of force begins to diminish, they are drawn into the vibrating conductor, where they disappear as lines of electric force and their energy is transformed into magnetic energy. This takes place in the following

manner: the lines of force in their tendency to contract undergo a lateral indentation or contraction in the direction of the z-axis (cf. figure 9); this lateral contraction advances until the given lines finally break, when the parts nearer the origin are drawn into the vibrating conductor and the outer parts are radiated off into space in the form of closed lines of force. The number of lines drawn into the vibrating conductor is always equal to that emitted by it, while the energy of the latter is of course diminished by that of the closed lines of force radiated off into space. The given vibration would therefore soon die away, if this loss of energy were not replaced by external forces brought to act on the vibrating conductor; in the present investigations we have assumed the presence of such external forces. To follow the course of the closed lines of force and their final radiation into space we extend our examination of the given system of curves to the periods $t=\tau, \frac{5\tau}{4}, \ldots$; for this purpose it is only necessary to reverse the directions of the arrows in the above figures. For $t=\tau$ the closed curves occupy the sphere R_4, figure 6, whereas the lines of force just emitted by the conductor have disappeared within it. For $t=\frac{5\tau}{4}$ new lines of force have already begun to shoot out of the conductor and force the closed lines of figure 6 into the outer region R_5 of figure 7. The subsequent course of these closed curves or lines of force, their occupation of the regions R_6 and R_7 of figures 8 and 9 respectively, and their gradual transformation into transverse waves or oscillations, as they recede from the origin, needs no further explanation. For a more dilate discussion of these systems of electric curves we refer the student to Hertz's original papers.*

*"*Die Kräfte electrischer Schwingungen, behandelt nach der Maxwell'schen Theorie*," v. 36, 1889, pp. 1-22; also "*Electric Waves by Hertz*," translated by Jones.

CHAPTER V.

SECTION XI. PECULIAR CHARACTER OF THE INTEGRALS THAT ARE TO BE INVESTIGATED.

THE determination of the general state of the ether, that is, of the successive stages through which it passes, due to an electromagnetic disturbance, meets with almost unsurmountable mathematical difficulties for even the simplest cases. That in many cases it is nevertheless possible to derive very simple laws from the equations is due to a favourable circumstance, the nature of which we shall first attempt to explain by means of a few illustrations from the theory of elasticity.

We suppose a stretched elastic band or cord with its one end A made fast, and its other end B set in motion at right angles to its initial position AB by the hand or some similar means, as a tuning fork. Generally speaking, this motion will give rise to a series of waves in the cord, and these upon reaching A will be reflected, so that after a short time the resulting waves will become very complicated, the motion of any particle of the cord being determined not only by the waves imparted to its end B, but by those reflected at both its ends. If the end B is made to move so slowly that the wave-length proves very large in comparison to the length of the cord, no trace of any waves will be detected. As the motion of the end B becomes more rapid, waves will begin to appear; the limit for the non-appearance of

waves in the cord is gradually passed, as the velocity of the end B attains to the same dimensions as those of the velocity of propagation of the waves; for as long as the velocity of the end B remains small in comparison to that of the propagation of the waves, any impulse—suppose the motion imparted to B to be in the form of a series of impulses—propagated along the cord will travel under uniform tension, since it will have sufficient time to adjust itself (its tension) before the arrival of a second impulse; consequently no variations in tension, that is, no waves will make their appearance in the cord. As soon, however, as the velocity of the end B increases to the dimensions of the velocity of propagation of the impulses, this will no longer remain the case, since the impulses imparted to the cord will follow in such rapid succession that an adjustment of the tension will no longer be possible.

However plain and simple the above considerations may seem, the mathematical conditions for the motion peculiar to the cord, when no waves appear, has not to my knowledge ever been thoroughly tested. This very kind of motion plays an important rôle, wherever masses distributed uniformly along lines, over surfaces, or throughout volumes, are set slowly in motion. This motion is not a stationary one, since its character can become entirely altered after a sufficiently long period; we could perhaps designate it as a motion approximately stationary or slow in comparison to the velocity of propagation of the waves themselves; an illustration of such a motion is that in the circuit of a battery that is left to itself and gradually runs down. In order to grasp this conception more firmly let us examine the simplest case: we denote by w any quantity that refers to the state of the cord at any point, for example, to the distance of that point from its initial position of rest or to its velocity, by a the velocity of propagation of the transverse waves along the cord, and by l its length. In order that the motion may be of the peculiar type just

postulated, the following condition must evidently hold for the impulse: the quantity

$$\frac{l}{a}\frac{dw_B}{dt},$$

where the index B denotes that the value of w for the end B is to be taken, must be small in comparison to the variations in the value of w, that is, it must be small in comparison to the difference between the largest and the smallest of these values; this we can express symbolically as follows:

$$\frac{l}{a}\frac{dw_B}{dt}\text{ sm. cp. } w_{\text{max.}} - w_{\text{min.}}. \quad\ldots\ldots\ldots\ldots\ldots(1)$$

If w is a length and $(w_{\text{max.}} - w_{\text{min.}})$ is of the same dimensions as l, we can write this condition in the simpler form

$$\frac{dw_B}{dt}\text{ sm. cp. } a.$$

If we characterize the motion of any particle by its distance from the fixed end A of the cord, we shall then always have

$$\frac{1}{a}\frac{dw}{dt}\text{ sm. cp. }\frac{w_{\text{max.}} - w_{\text{min.}}}{x} = \frac{dw}{dx}. \quad\ldots\ldots\ldots\ldots(2)$$

The wave-motion w is now given by the familiar function

$$w = \text{f}(x \pm at); \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

by its differentiation with regard to t and x we have

$$\frac{dw}{dt} = \frac{d\text{f}(x \pm at)}{d(x \pm at)} \cdot \frac{d(x \pm at)}{dt} = \pm \text{f}' . a.$$

and $$\frac{dw}{dx} = \frac{d\text{f}(x \pm at)}{d(x \pm at)} \cdot \frac{d(x \pm at)}{dx} = \text{f}',$$

from which it follows that

$$\frac{1}{a}\frac{dw}{dt} = \pm\frac{dw}{dx}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Comparing this equation (4) for wave-motion with condition (1), we recognize that the latter is indeed the condition for the required slow motion, for it expresses the fact that the velocity with which w changes for this motion is much smaller than the rate of change of w for perceptible wave-motion.

If we denote by y the distance of any particle of the cord from its initial position of rest, its motion is given by the following well-known partial differential equation:

$$\frac{1}{a^2}\frac{d^2y}{dt^2}=\frac{d^2y}{dx^2}. \qquad \text{(5)}$$

We next replace w by y, in the condition (2) for the slow motion, differentiate with regard to t and x and we find

$$\frac{1}{a}\frac{d^2y}{dt^2} \text{ sm. cp. } \frac{d^2y}{dt\,dx}$$

and

$$\frac{1}{a}\frac{d^2y}{dt\,dx} \text{ sm. cp. } \frac{d^2y}{dx^2},$$

from which conditions it follows that

$$\frac{1}{a^2}\frac{d^2y}{dt^2} \text{ sm. cp. } \frac{d^2y}{dx^2}.$$

We see, therefore, that for slow motion $\frac{1}{a^2}\frac{d^2y}{dt^2}$ can be rejected in comparison to $\frac{d^2y}{dx^2}$, and hence that equation (5) can be written

$$\frac{d^2y}{dx^2}=0,$$

the integral of which is

$$y=x\phi(t)+\psi(t).$$

Since for $x=0$, $y=0$, and for $x=l$, y is a given function f of t, it follows that

$$y=\frac{x}{l}\mathrm{f}(t), \qquad \text{(6)}$$

the equation of a straight line, which corresponds to a uniform tension along the cord.

The above calculation is only an approximate one, a first approximation; second or higher approximations can therefore be found by well-known methods: we denote, namely, the value (6) already found for y by y_1, put

$$y=y_1+y_2=\frac{x}{l}\mathrm{f}(t)+y_2$$

in equation (5) and we find

$$\frac{1}{a^2}\frac{x}{l}\frac{d^2}{dt^2}[\mathrm{f}(t)]+\frac{1}{a^2}\frac{d^2y_2}{dt^2}=\frac{d^2y_2}{dx^2};$$

for reasons similar to the above $\frac{1}{a^2}\frac{d^2y_2}{dt^2}$ can be rejected in comparison to $\frac{d^2y_2}{dx^2}$, and this equation can be written

$$\frac{1}{a^2}\frac{x}{l}\mathrm{f}''(t)=\frac{d^2y_2}{dx^2},$$

the integral of which is

$$y_2=\frac{x^3}{6a^2l}\mathrm{f}''(t)+x\phi(t)+\psi(t),$$

where ϕ and ψ are to be determined by the conditions that for $x=0$ and $x=l$, $y_2=0$; we have then

$$\psi(t)=0 \quad\text{and}\quad 0=\frac{l^2}{6a^2}\mathrm{f}''(t)+l\phi(t),$$

hence
$$\phi(t)=-\frac{l}{6a^2}\mathrm{f}''(t).$$

As a second approximation for this integral of equation (5) we find therefore

$$y=\frac{x}{l}\mathrm{f}(t)+\left(\frac{x^3}{6a^2l}-\frac{xl^2}{6a^2}\right)\mathrm{f}''(t). \quad\text{..............(7)}$$

The repeated substitution in equation (5) of the last approximate value for y gives a series for y; its convergence is, however, limited, but as long as it converges it will agree with the solution of equation (5) by arbitrary functions and also with its general integral. Analogous series also occur in the theory of electromagnetism, for examples, those for the electrostatic action of variable electric currents, and the ponderable action of magnets which are known to gradually loose their intensity when left to themselves.

It is evident that the second and all subsequent approximations do not represent the equation of a straight line, but slight deviations from it, and that the cord can thus no longer be considered as absolutely uniformly stretched or under uniform tension, as is the case for the first approximation (6). The deviation of the equation for the cord from that of a straight line will in general become greater, as the number of terms of the series is increased. This motion, expressed by such approximations as (6) and (7), is now the kind of motion that we shall characterize as our *slow* motion.

In relation (2) the differential quotients of one and the same quantity w are compared with each other. In order to compare the differential quotients of two different quantities u and v, it is only necessary to bear in mind that during the motion kinetic energy is continually being transformed into work, and conversely work into kinetic energy, that the increment of the kinetic energy and that of the work done will therefore be quantities of the same order, and hence that the differential quotients of these quantities will admit of a comparison similar to that of the differential quotients of a single variable w. As illustration of this let us again consider the stretched elastic band AB; suppose, however, that its free end B is made to vibrate longitudinally instead of transversely, as in the previous case. Let ξ be the longitudinal displacement of any particle C of the cord towards B, and ξ_B its value for $x=AB=l$. If p denotes the elastic

force acting on its unit-cross-section and E its modulus of elasticity, we know then from the theory of elasticity that

$$p = E\frac{d\xi}{dx}. \quad \text{.........................(8)}$$

The kinetic energy of a piece dx of the cord is

$$\frac{\rho q\, dx}{2}\left(\frac{d\xi}{dt}\right)^2, \quad \text{.........................(9)}$$

where q denotes its cross-section and ρ its density. The work done δA per volume-element $q\,dx$ by the elastic forces during the deformation $\delta\xi$ is evidently

$$\delta A = p[\delta\xi(x+dx) - \delta\xi(x)]q\,dx = pq\delta\left(\frac{d\xi}{dx}\right)dx,$$

which by the above values (8) for p can be written

$$\delta A = E\left(\frac{d\xi}{dx}\right)\delta\left(\frac{d\xi}{dx}\right)q\,dx;$$

hence the total work done during the deformation ξ will be

$$A = q\,dx\,E\int\frac{d\xi}{dx}\delta\left(\frac{d\xi}{dx}\right) = \frac{q\,dx}{2}E\left(\frac{d\xi}{dx}\right)^2. \quad \text{.........(10)}$$

The variation of $q\,dx$ during this deformation has been neglected, since $\delta(q\,dx)$ is a quantity, whose order of magnitude is lower than that of those considered.

As the expressions (9) and (10) are being continually transformed into each other, their increments must be of the same order. Hence if we put

$$u = \frac{d\xi}{dt}\sqrt{\rho/E}, \quad v = \frac{d\xi}{dx}, \quad \text{................(11)}$$

u and v will be quantities of the same order.

For the slow motion (cf. relation (2)) $\frac{1}{a}\frac{du}{dt}$ must now be

small in comparison to $\frac{du}{dx}$ and $\frac{dv}{dx}$, since u and v are of the same order; it follows therefore that

$$\frac{1}{a}\frac{du}{dt} \text{ sm. cp. } \frac{dv}{dx}. \quad\text{.....................(12)}$$

The equation of motion for the longitudinal vibrations of the cord

$$\frac{d^2\xi}{dx^2}=\frac{1}{a^2}\frac{d^2\xi}{dt^2}=\frac{\rho}{E}\frac{d^2\xi}{dt^2},$$

which written in the above notation (11) is

$$\frac{dv}{dx}=\frac{1}{a}\frac{du}{dt},$$

thus reduces to the following for the required motion

$$\frac{dv}{dx}=\frac{d^2\xi}{dx^2}=0,$$

as above.

SECTION XII. APPLICATION OF THE PRINCIPLES OF PRECEDING ARTICLE TO AERODYNAMICS AND ELECTRICITY (ASONIC AND APHOTIC MOTIONS).

The illustrations of the previous article may appear to be of almost childish simplicity; they are, however, the most complete analogies to the solutions of the fundamental equations of electro-magnetism that are presently to be considered. On account of the enormous velocity of propagation of all disturbances through the ether, all means of exciting electro-magnetic phenomena yet known to us give rise to motions of the peculiar character described in the preceding article; the only exceptions are the phenomena of light and the electric oscillations discovered by Hertz. Even the latter at

distances from the vibrator that are small in comparison to their wave-length are also of this character, as we have seen in § 10; this appears moreover plausible from a consideration of the analogous case for the vibrating cord, namely, when its length is small in comparison to its wave-length.

An exhaustive analysis of the nature of this slow motion would lead us too far from our subject. In the first place we should have to supplement relation (1), which characterizes the given impulse, by the condition that it be maintained for only a short time with a period of vibration, that is an exact multiple of that belonging to the medium, to which this impulse has been imparted, for we should then have an exceptional case, especially if we excluded all damping of the waves (compare the analogous case of two cords vibrating with multiple periods).

Before returning to the electro-magnetic phenomena, let us examine one more case of slow motion, namely, that of a gas, within which the pressure p (per unit-surface) and density ρ are given by the law: $p = C\rho^n$; if u, v, w are the components of its velocity at any point, then

$$\rho\frac{(u^2+v^2+w^2)}{2}$$

will be its kinetic energy per unit-volume. The velocity with which any motion is propagated through the given gas is now

$$a=\sqrt{\frac{dp}{d\rho}}=\sqrt{nC\rho^{n-1}}.$$

The work done by its expansion (contraction) per unit-volume will be

$$\int p d(\rho_0/\rho)=\rho_0\int p d(1/\rho)=\frac{\rho_0 C}{n-1}\rho^{n-1}+\text{const.},$$

where ρ_0 denotes its initial density.

The increments of ρu^2 and $\frac{\rho_0 C}{n-1}\rho^{n-1}$ will therefore be quantities of the same order, since the work done is continually being transformed into kinetic energy and kinetic energy into work during the passage of any disturbance; hence the increments of the square-roots of these quantities will be of the same order or, if we assume that ρ_0 and ρ are of the same order—no other assumption would be plausible—the increments of

$$\rho u \quad \text{and} \quad \rho\sqrt{\frac{C}{n-1}\rho^{n-1}},$$

or, if $n/n-1$ is finite, those of ρu and ρa. For $n=1$, where the above proof is no longer valid, special investigations would be necessary; these we shall, however, omit here.

If we put

$$a\rho=k, \quad \rho u=l, \quad \rho v=m, \quad \rho w=n,$$

the variations in all these quantities will then be of the same order. The equation of continuity for a gas is now

$$\frac{1}{a}\frac{dk}{dt}+\frac{dl}{dx}+\frac{dm}{dy}+\frac{dn}{dz}=0.$$

For slow motion the first term of this equation may therefore, according to the above principles, be rejected in comparison to the other terms; we have then

$$\frac{dl}{dx}+\frac{dm}{dy}+\frac{dn}{dz}=0,$$

whose integral would be a first approximation. If initially the density of the given gas is everywhere the same, and if the envelope within which it is contained, and any body immersed in it for the purpose of setting it in motion, move in such a manner that its volume does not change, its density will then also remain approximately the same throughout, and its motion will

thus be similar to that of an incompressible fluid. If, on the other hand, its volume changes—imagine, for instance, that the gas is contained in a cylinder, one end of which is closed by a movable piston—the variations in density will also be slow, and will remain very nearly equal throughout the entire gas.* In each case the density of the gas will therefore remain approximately uniform throughout. A sufficiently rapid impulse can of course give rise to waves similar to those in an incompressible fluid. We must therefore conclude that the above motion (variations in density) will be slow in comparison to the diffusion of the sound-waves through it. This kind of motion we shall designate by the word *asonic*.

Such asonic motions can and generally will be vibratory; for example the transverse vibrations of a stretched cord made fast at both ends and weighted in the middle by a mass, that is very small in comparison to its own mass, or the torsional vibrations of a wire suspended by its one end and supporting at its other a mass of large moment of inertia will possess the above peculiarities of asonic motion, for these motions, not only the vibrations themselves, but the variations in tension of the cord or wire will be slow in comparison to the velocities with which they are transmitted.

The above also holds for the ether. We shall now designate those motions of the ether produced by slow impulses, whereby light-waves and similar phenomena must be excluded, by the word *aphotic*, where we are of course to understand by φῶς not only those vibrations of the ether that are susceptible to the retina of the eye, but also Hertz's oscillations and heat rays. Still slower oscillations than those of the latter (φῶς) class are

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* In the equation of hydrodynamics the variations in ρu^2 and $p = C\rho^n$ are of the same order, so that the variations in ρ, but not those in p can be neglected. Consequently the equations of motion will remain unaltered.

indeed possible, for example, the oscillating discharges of condensers (Leyden jars); it is, however, impossible to measure the waves generated by these discharges in places of the dimensions of our laboratories or even indeed of our earth.

We have already remarked that with the exception of the phenomena of light, heat rays, and Hertz's oscillations, all means of exciting electro-magnetic phenomena known to us satisfy relation (2); this is due to the enormous velocity of propagation of the electro-magnetic disturbances, all of which, with the above exceptions, can therefore be looked upon as aphotic motions of the ether, where all traces of the wave-motion of electro-magnetism remain hidden to our observation.

In order to find the relations analogous to (12) for the ether we must bear in mind that, if there is to be any appreciable interchange between the quantities P, Q, R and α, β, γ, the principle of the conservation of energy must hold, that is, dT and dV (cf. equations (7, II.) and (8, II.)) must be quantities of the same order; consequently that the increments of the separate terms on the right of these equations (7, II.) and (8, II.), and hence those of

$$P\sqrt{D},\ Q\sqrt{D},\ R\sqrt{D},\ \alpha\sqrt{M},\ \beta\sqrt{M},\ \gamma\sqrt{M}$$

must be quantities of the same order. As D and M are finite numbers, the increments of P, Q, R, α, β, γ must likewise be quantities of the same order; the latter play the rôle of the variables u and v in relation (12).

$\mathfrak{V}$ is the velocity of propagation of the electric waves in air. In conformity to relation (12) we can therefore reject the left-hand members of equations (10, II.) in comparison to those on the right; we have then

$$\frac{dR}{dy}=\frac{dQ}{dz},\quad \frac{dP}{dz}=\frac{dR}{dx},\quad \frac{dQ}{dx}=\frac{dP}{dy}. \quad \text{..........(13)}$$

If on the other hand there is no appreciable interchange between the quantities P, Q, R and α, β, γ, every

term of equations (10, II.) that contains any one of the latter three quantities must vanish *eo ipso*; we obtain then the same equations (13) as above.

From equations (13) it follows that

$$P=-\frac{d\phi}{dx},\quad Q=-\frac{d\phi}{dy},\quad R=-\frac{d\phi}{dz}.\ \dots\dots\dots(14)$$

We require only one more equation for the determination of ϕ; we obtain this most readily by the elimination of α, β, γ from equations (9, II.), which gives

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]$$
$$+\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}=0,\dots\dots(15)$$

an equation already found in articles 5 and 6. We have then four equations (14) and (15) in P, Q, R—α, β, γ have already been eliminated from these equations—for the determination of these quantities as functions of ϕ, to which we shall proceed in the next article.

For aphotic motion the left-hand sides of equations (9, I.) vanish in conformity to the above conditions, and we thus obtain the following equations for large values of L:

$$\left.\begin{aligned}\frac{d\beta}{dz}-\frac{d\gamma}{dy}&=\frac{4\pi L}{\mathfrak{V}}(P+X)\\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz}&=\frac{4\pi L}{\mathfrak{V}}(Q+Y)\\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}&=\frac{4\pi L}{\mathfrak{V}}(R+Z)\end{aligned}\right\}\dots\dots\dots\dots\dots(16)$$

If, on the other hand L is small, these equations reduce to

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}=\frac{d\alpha}{dy}-\frac{d\beta}{dx}=0.$$

The investigations of this article and similar ones to be frequently introduced in the following are, of course, entirely superfluous for those accustomed to Kirchhoff's method of first finding particular integrals of our fundamental equations and then interpreting them. This method of investigation is surely no more lucrative than ours, provided we show that the given particular integrals, whose physical meaning and importance we first demonstrate according to our above method, satisfy our differential equations exactly, that is, without the rejection of any quantities whatever (cf. § 16).

CHAPTER VI.

SECTION XIII. CONCEPTION OF THE FREE ELECTRICITY.

By equations (14, V.) we can write equation (1, III.) in the form

$$\epsilon_r = -\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)\right]. \quad \ldots.(1)$$

If equations (14, V.) are fulfilled, all the forces with the exception of the external electromotive forces—that is, those that act apparently at a distance in our concrete representation—which act on unit-quantity of electricity will be given by formula (12, III.); they are namely,

$$-\frac{d\phi}{dx}, \quad -\frac{d\phi}{dy}, \quad -\frac{d\phi}{dz};$$

ϕ is then called the electrostatic potential. It is now customary to measure ϕ in all systems of units in such a manner that its negative derivatives with regard to the coordinates give exactly (without any factor of proportionality) the forces that act on unit-quantity of electricity, so that equations (14, V.) remain unaltered in all systems of units. If in any other system of units $P_h = hP$, the value of ϕ measured in this new system—let us denote its value by ϕ_h—must be taken equal to $h\phi$; on the other hand, according to formula (1, III.) it is evident that

$$\epsilon = h\epsilon_h. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If the given body is homogeneous, that is, if D, etc., are constant, we find the following value for ϵ_h by equation (1):

$$\epsilon_h = -\frac{D}{4\pi h}\nabla^2\phi = -\frac{D}{4\pi h^2}\nabla^2\phi_h. \qquad \text{.............(3)}$$

Compare also formula (18, III). If the body is non-homogeneous, we have

$$\epsilon_h = -\frac{D}{4\pi h^2}\nabla^2\phi_h - \frac{1}{4\pi h^2}\left(\frac{d\phi_h}{dx}\frac{dD}{dx} + \frac{d\phi_h}{dy}\frac{dD}{dy} + \frac{d\phi_h}{dz}\frac{dD}{dz}\right). \quad (4)$$

From formula (3) it follows then that

$$\nabla^2\phi_h = -\frac{4\pi h^2}{D}\epsilon_h,$$

which for $h=1$ reduces to

$$\nabla^2\phi = -\frac{4\pi}{D}\epsilon.$$

By formulae (1) and (14, V.) this last relation can be written

$$\nabla^2\phi = -\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}\right). \qquad \text{..............(5)}$$

We next make use of the following theorem. If $\nabla^2\Omega = -4\pi\sigma$, where σ has a given finite value at every point of space, and Ω vanishes at infinite distance, moreover if Ω and its derivatives $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}, \frac{d\Omega}{dz}$ are single-valued and continuous throughout space, Ω is then determined uniquely, being the Newtonian potential function of a mass of the density σ.* All hese conditions evidently hold for ϕ. That $\phi, \frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$ are single-valued and continuous throughout space is

* Kirchhoff's "*Vorlesungen über die mathematische Physik,*" p. 189-193.

evident from the nature of the problem, all possible doubts having been removed once for all by the assumption of the principle of the continuity of transitions (cf. § 2). ϕ can also always be so chosen that it will vanish at infinity, for since we have assumed that all electro-magnetic disturbances become imperceptible at infinite distance, that is, that P, Q, R or the derivatives of ϕ with regard to the coordinates (cf. formulae (14, V.)) vanish at these limits, we can choose the arbitrary constant that appears by the integration of equations (14, V.), since it has no effect whatever on our results, in such a manner that ϕ also vanishes at infinity. We see, therefore, that ϕ can be conceived as the Newtonian potential function of a mass, whose density at any point (x, y, z) is determined by the value of the expression

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) \qquad (6)$$

at that point. From the theory of the potential we know then that

$$\phi(x_1, y_1, z_1)=\frac{1}{4\pi}\sum\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\frac{1}{\rho}, \qquad (7)$$

where ρ denotes the distance between the point (x_1, y_1, z_1), at which the value of ϕ is sought, and any point (x, y, z) of space, and the summation is to be extended to all points at which the expression (6) assumes a value different from zero.

From the differentiation of equations (14, V.) it follows that

$$\left.\begin{aligned}\nabla^2 P&=\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\\ \nabla^2 Q&=\frac{d}{dy}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\\ \nabla^2 R&=\frac{d}{dz}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\end{aligned}\right\} \qquad (8)$$

Since P, Q, R and their derivatives with regard to the coordinates are single-valued and continuous, and the former vanish at infinite distance, the above theorem can also be applied to these equations (8), and these quantities likewise be regarded as Newtonian potential functions, and thus written

$$P(x_1, y_1, z_1) = -\frac{1}{4\pi}\sum\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\frac{1}{\rho},$$

$$Q(x_1, y_1, z_1) = -\frac{1}{4\pi}\sum\frac{d}{dy}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\frac{1}{\rho},$$

$$R(x_1, y_1, z_1) = -\frac{1}{4\pi}\sum\frac{d}{dz}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\frac{1}{\rho},$$

Comparing these expressions with that (7) for ϕ we see that

$$P = -\frac{d\phi}{dx}, \quad Q = -\frac{d\phi}{dy}, \quad R = -\frac{d\phi}{dz},$$

a confirmation of equations (14, V.), namely, that the values of P, Q, R at any point (x_1, y_1, z_1) are in fact the derivatives with regard to the coordinates of the above value (7) for $\phi\ (x_1, y_1, z_1)$.

The theory of the potential thus offers a complete key to the solution of all problems of the above kind; this is indeed always the case, when the given quantities are the partial derivatives of a function with regard to the coordinates (compare the so-called velocity-potential in fluid-motion).

In spite of the inconvenience caused by the introduction of an arbitrary constant, whose value will of course have no effect on our results, it will nevertheless be found advisable later to multiply the above expression (6) by such an arbitrary constant $\mathfrak{e}$; this can be interpreted by conceiving all space to be filled with a mass whose density ϵ_f at any point is

$$\epsilon_f = \frac{\mathfrak{e}}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) \quad \text{.................(9)}$$

This expression treated similarly to equation (5) gives

$$\epsilon_f = -\frac{\mathfrak{k}}{4\pi}\nabla^2\phi. \qquad\qquad (10)$$

$\mathfrak{k}$ shall moreover have the same value at every point of space and during all periods. ϕ thus becomes the potential of a mass whose density is $\epsilon_f/\mathfrak{k}$ and whose value at any point of space is thus given by the following integral:

$$\phi(x_1, y_1, z_1) = \frac{1}{\mathfrak{k}}\int\frac{\epsilon_f d\tau}{\rho}, \qquad\qquad (11)$$

where $d\tau$ is any volume-element of this fictitious mass and ϵ_f the value of the expression

$$\frac{\mathfrak{k}}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) = -\frac{\mathfrak{k}}{4\pi}\nabla^2\phi$$

at the given point. The integration of formula (11) is to be extended to the volume-elements of all the bodies under electric stress, that is, to the confines of space, since $\epsilon_f = 0$ wherever the medium is not strained. We shall call ϵ_f the *density of the free electricity* at the given point. If the expression for ϵ_f is negative at given points of space, as is often the case, it only denotes that the fictitious mass at those points repels instead of attracts, according to the inverse square of the distance; such repulsions are, for example, those between so-called similar electricities.

For the atmosphere $D = 1$. Hence, if we put here $\mathfrak{k} = 1$, it follows that for this important dielectric the free electricity becomes identical to the real. To find a simple relation between ϵ_f and ϵ_r for other dielectrics, we shall, however, be obliged to make some special hypothesis concerning their behaviour, that is, that of the electric polarization or induction. We shall consider this subject in the next article.

For the most general case where, namely, equations (13, V.) do not hold, we can nevertheless retain the con-

ception of the free electricity by defining it similarly to the real, namely, by putting

$$\epsilon_f = \frac{\mathfrak{k}}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right), \quad\text{..................(12)}$$

and retaining the above definition (5) for ϕ. We can then write P, Q, R in two parts, as follows:

$$P = -\frac{d\phi}{dx}+P_1, \quad Q = -\frac{d\phi}{dy}+Q_1, \quad R = -\frac{d\phi}{dz}+R_1, \quad\text{...(13)}$$

and designate $\quad -\dfrac{d\phi}{dx}, \quad -\dfrac{d\phi}{dy}, \quad -\dfrac{d\phi}{dz}$

as the components of the electrostatic force—these always have a potential—and P_1, Q_1, R_1 as those of the electrodynamic force.

If we replace P, Q, R by their values (13) in the expression

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}.$$

we have

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz} = -\nabla^2\phi+\left(\frac{dP_1}{dx}+\frac{dQ_1}{dy}+\frac{dR_1}{dz}\right),$$

from which and the definition (5) for ϕ it follows that

$$\frac{dP_1}{dx}+\frac{dQ_1}{dy}+\frac{dR_1}{dz}=0; \quad\text{................(14)}$$

this corresponds to the fact that the electrodynamic forces P_1, Q_1, R_1 have no potential.

The above division (13) of P, Q, R will be of great value to us whenever the given motion approaches in any degree the aphotic motion, since the series analogous to series (7, V.) will then converge rapidly; this division of the electric forces is, however, often a matter of much uncertainty, and in all cases where the electrodynamic forces are large in comparison to the electrostatic, as in all rapid disturbances (oscillations) of the ether, it will be of little use, since the given series will diverge.

SECTION XIV. THIRD FEATURE OF OUR CONCRETE REPRESENTATION; CONCEPTION OF THE ELECTRIC POLARIZATION OR INDUCTION.

The conception of the free electricity would be of little value, unless we were able to connect it by some good descriptive illustration with that of the real electricity; this new illustration shall form the third feature of our concrete representation. We conceive, namely, that the electric fluids of insulators, though bound, are nevertheless capable of displacements within their separate volume-elements; the state into which the ether is thrown by these displacements we shall call that of *polarization* or *induction*.

Let us consider any volume-element $dx\,dy\,dz$ of a given insulator. We conceive then that as soon as electric forces are brought to act on this element its electric fluids are displaced within it, and in such a manner that the amount of positive fluid that collects on its $dy\,dz$ side, which faces the positive x-axis, is

$$\frac{D-\mathfrak{k}}{4\pi}P\,dy\,dz,$$

whereas the same amount of negative fluid appears on its opposite $(dy\,dz)$ side. As in the theory of magnetism the expression $a\,dy\,dz\,.\,dx$ is called the magnetic moment (per volume-element) along the x-axis, we shall designate here the quantity

$$\frac{D-\mathfrak{k}}{4\pi}P\,dy\,dz\,.\,dx=\mathfrak{x}\,dx\,dy\,dz\;\ldots\ldots\ldots\ldots(15)$$

as the *electric* moment (per volume-element) along the x-axis, $\mathfrak{x}$ being its electric moment per unit volume. The action at a distance of such a polarized volume-element like that of a magnet will of course only depend on this moment.

Similar expressions shall hold for the y and z-axes. The displacements in a neighbouring volume-element are found by replacing x, y, and z in these expressions by $x+dx$, $y+dy$, and $z+dz$ respectively. The amount of negative fluid displaced within the volume-element $x+dx, y, z$ and appearing on its $dydz$ side will be then

$$\frac{D-\mathfrak{e}}{4\pi}P(x+dx, y, z)dy\,dz=\frac{D-\mathfrak{e}}{4\pi}\left(P+\frac{dP}{dx}dx\right)dy\,dz;$$

the surplus of positive fluid or electricity on this side will therefore be the algebraic sum of the fluids withdrawn from the volume-elements x, y, z and $x+dx, y, z$ and appearing on it, namely,

$$\frac{1}{4\pi}\left(\mathfrak{e}-\frac{d(DP)}{dx}\right)dx\,dy\,dz.$$

Similar expressions will hold for the sides $dz\,dx$ and $dx\,dy$ of the given volume-element. The surplus of electricity in any volume-element $dxdydz$ brought about by electric polarization or induction will therefore be

$$\frac{\mathfrak{e}}{4\pi}\left[\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right]dx\,dy\,dz$$

$$-\frac{1}{4\pi}\left(\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right)dx\,dy\,dz=\epsilon_p dx\,dy\,dz.\dots(16)$$

We shall call ϵ_p the *density of the electricity due to electric polarization or induction*.

In our concrete representation we shall assume that the electric fluids act at a distance. Since the amount of the positive fluid not displaced or, as we shall say, bound, is the same as that of the bound negative fluid, the only electricities that can act at a distance will be the real and that due to electric polarization; we shall call their sum the free electricity. Its density is then $\epsilon_f=\epsilon_r+\epsilon_p$. Substituting here for ϵ_r and ϵ_p their values (1, III.) and (16) respectively, we find equation (12); we see therefore that the new feature of our concrete

representation exactly illustrates the relation that exists between the real and free electricities.

The appearance of the free electricity due to electric polarization causes an equal amount of neutral electricity of the opposite kind (sign) to become inactive. We shall call this inactive electricity the *bound electricity*, and denote its density by ϵ_b. We have then

$$\epsilon_b = -\epsilon_p.$$

We shall next assume in our concrete representation that the electric displacements which give rise to the electric polarization are likewise produced by the forces P, Q, R, that act on the electric fluids of § 8. Let us first consider the case where no external electromotive forces are active; according to formula (12, III.) the force that acts along the positive x-axis on unit-quantity of the positive fluid is P. In our concrete representation we must therefore assume that

$$\mathfrak{x} = \frac{D - \mathfrak{e}}{4\pi} P = \epsilon_{vH} P, \quad \ldots\ldots\ldots\ldots\ldots\ldots(17)$$

that is, that the electric moment per unit-volume is proportional to and in the direction of the force that acts on unit-quantity of electricity. In order that our notation may be as similar as possible to that of von Helmholtz, we have designated the factor of proportionality by ϵ_{vH}, whereas von Helmholtz* denotes it by ϵ. Stefan† calls this factor the constant of electric polarization and denotes it by h.

We can conclude the new feature of our concrete representation by assuming that the electric displacement gives rise to molecular forces equal but opposite in direction to the component-forces P, Q, R that produce it, the electric displacement advancing until the

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* *Wissenschaftliche Abhandlungen*, v. i., p. 616.

† *Wiener Berichte*, 70, p. 634, 1874.

molecular forces become equal to the electric forces P, Q, R, when equilibrium sets in.

The work necessary to bring about the required electric polarization in our concrete representation can be found as follows: suppose that, when $\mathfrak{x}$ increases by $d\mathfrak{x}$, the quantity of positive fluid displaced from the one side of the given volume-element to its opposite side is $d\mathfrak{x}\, dy\, dz$, the same quantity of negative fluid being thereby liberated on its other side; this quantity of fluid is displaced along the path dx, and the force acting on it is $P d\mathfrak{x} dy dz$; hence the work done by this component force will be

$$P\, d\mathfrak{x}\, dx\, dy\, dz = \frac{\mathfrak{x}\, d\mathfrak{x}}{\epsilon_{vH}} dx\, dy\, dz,$$

and similarly those done by the other two component forces will be

$$Q\, d\mathfrak{y}\, dx\, dy\, dz = \frac{\mathfrak{y}\, d\mathfrak{y}}{\epsilon_{vH}} dx\, dy\, dz$$

and

$$R\, d\mathfrak{z}\, dx\, dy\, dz = \frac{\mathfrak{z}\, d\mathfrak{z}}{\epsilon_{vH}} dx\, dy\, dz.$$

The total work done per unit-volume against the molecular forces, that is, that necessary to bring about the required electric polarization $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ will therefore be

$$\int \frac{1}{\epsilon_{vH}} (\mathfrak{x}\, d\mathfrak{x} + \mathfrak{y}\, d\mathfrak{y} + \mathfrak{z}\, d\mathfrak{z}) = \frac{\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2}{2\epsilon_{vH}} \quad \ldots\ldots\ldots\ldots\ldots (18)$$

$$= \frac{\epsilon_{vH}}{2}(P^2 + Q^2 + R^2) = \frac{D - \mathfrak{k}}{8\pi}(P^2 + Q^2 + R^2).$$

We should always bear in mind that all these conceptions are to serve only as descriptive illustrations of results, all of which either are comprised in the fundamental assumptions and equations developed in the first chapter, or follow necessarily from them. They are therefore by no means to be regarded as new assumptions, but merely as new descriptive representa-

tions, by which we are enabled to illustrate the various consequences that can be derived from our fundamental equations.

Since no electric polarization has yet been confirmed in the metals, we can assume the same value for D for all metals and then put our arbitrary constant $\mathfrak{e}$ equal to that value; no displacements of the electric fluids would then be possible within their separate volume-elements, and the third feature of our concrete representation would thus agree in every respect with empirical results. The fact that water has been asserted on the one hand to be susceptible to electric polarization, and, on the other, to possess no metallic conductivity—we overlook its electrolytic conductivity—would surely not justify us in denying all bodies capable of electric polarization every metallic conductivity, and in thus concluding that the same non-electric medium should answer for all bodies from the non-conductors to the metals exclusively. Such a conclusion from or specializing of Maxwell's theory would at least be forced; moreover it would be exposed to constant refutation by experiment.

It would therefore be better to assume that the metals are also capable of electric polarization. This can be accomplished in two ways: first, by conceiving only one type* of electricity in the given conductor, and assuming that every one of its particles undergoes a displacement equal to the sum of the displacements which the given particle would undergo in consequence of its conductivity and its electric polarization alone; and, secondly, by assuming two types* of electricity in the given conductor, the one the so-called current electricity, which produces by its flow the phenomena of electric conduction and behaves exactly like the electric fluids of Chapter III., and the other, the polarized electricity,

* "Type" and "kind" are not to be interchanged; "kind" refers to the sign, positive or negative, of the given fluid.

due to displacements of electric fluids similar to those in dielectrics or non-conductors. Since, according to the dualistic theory, each type of electricity is to be conceived as a mixture of equal quantities of positive and negative fluids, only the surplus of the one fluid over the other comes into consideration here. Although the second convention or conception is more complicated than the first, we shall nevertheless adopt it here, chiefly on account of its greater conformability, for we can then assume that the external forces only act on the electric fluids of the first type and not on those of the second, and thus retain formulae (17) and (18) for the electric polarization instead of accepting the more general expressions

$$x = \epsilon_{vH}(P+X) \quad \text{and} \quad \frac{x^2}{\epsilon_{vH}} = \frac{D-\mathfrak{e}}{4\pi}(P+X)^2.$$

The general assumption that the external electromotive forces also act on the electric fluids of the second type can differ from the above assumption only in regions where X, Y, Z do not vanish; on the other hand, little is known of such regions, for the apparent action at a distance of regions where hydro or thermo-electromotive forces, or electromotive forces arising from friction, reside, have never been observed. It is possible, however, that Maxwell's theory is deficient in this respect. We could assume that the fluids of the first type are displaced by the chemical migration of the ponderable atoms, and that these displacements alone give rise to the hydro-electromotive forces, whereas the fluids of the second type are displaced by some other forces, and thus give rise to no external electromotive forces, and hence conversely that in regions where X, Y, Z do not vanish the external electromotive forces have no effect whatever on the electric polarization. Here we shall, however, retain our above assumption concerning X, Y, Z, namely, that they only act on the electric fluids of the first type and according to the law given above, and form our concrete

representation in such a manner that it reproduces the given equations. We shall return to this subject in § 36, where we shall become acquainted with a substitution that brings about a symmetrical distribution of the external electromotive forces between the current electricity and that due to electric polarization, that is between the quantities

$$\frac{4\pi L}{\mathfrak{B}}P \text{ and } \frac{D}{\mathfrak{B}}\frac{dP}{dt}.$$

According to equations (12, III.), the components of the force that acts on unit-quantity of neutral electricity are P, Q, R, provided no external electromotive forces are acting. If equations (14, V.) hold, P, Q, R will have a potential ϕ, which can be conceived as the Newtonian potential of a mass of the density $\epsilon_f/\mathfrak{k}$; it follows therefore that the force that acts on the quantity e_n of neutral electricity will be equivalent to the force obtained from the assumption that every quantity e_f of free electricity acts on this quantity e_n of neutral electricity according to the law

$$\frac{e_f e_n}{\mathfrak{k}\rho^2}, \qquad \text{(19)}$$

where ρ denotes the distance between e_f and e_n.

SECTION XV. E_r AND E_f FOR APHOTIC MOTION.

According to the principle of the continuity of transitions, we have conceived the dividing surface between adjoining bodies to be a very thin transition-film, within which all quantities vary continuously. We must therefore comprise in integral (11) all the volume-elements of all the transition-films. Since P, Q, R generally undergo very rapid changes as we pass through these films, their volume-elements though few in number will nevertheless contribute a finite amount to integral (11);

it is now often desirable to know this amount, arising from the transition-films, separately from that which the interiors of the given bodies contribute to our integral. In order to denote that the given integration is to be extended not only to all volume-elements of the interiors of the given bodies, but also to all volume-elements of their transition-films, we shall suffix the index U to its integral sign; if, on the other hand, the integration is only to be extended to the volume-elements of their interiors, we shall suffix the index T. We can then write formula (11) as follows:

$$\phi=\frac{1}{\mathfrak{k}}\int\limits_{U}\frac{\epsilon_f d\tau}{\rho}=\frac{1}{\mathfrak{k}}\left[\int\limits_{T}\frac{\epsilon_f d\tau}{\rho}+\int\frac{E_f do}{\rho}\right]. \qquad (20)$$

The surface-density E_r of the real electricity on such a dividing surface—this is equal to the quantity of real electricity contained in the volume-element δdo—has already been found on page 53; it is, namely,

$$E_r=\frac{1}{4\pi}(D_1P_1-D_0P_0),$$

when the x-axis coincides with the normal to the surface at the given point, or

$$E_r=\frac{1}{4\pi}(D_1N_1-D_0N_0), \qquad (21)$$

when the given normal makes the angles (n, x), (n, y), (n, z) with the coordinate axes. N is the component of the vector (P, Q, R) along the normal n to do, where the normal is to be drawn from the side, to which the index 0 refers, towards that which has the index 1.

As we have defined the total amount of real electricity as $\int\epsilon_r d\tau$, we shall next define the total amount of free electricity as $\int\epsilon_f d\tau$. The surface-density E_f of the free electricity at any point will then be

$$E_f = \int \epsilon_f dn = \frac{\mathfrak{e}}{4\pi}\int\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)dn,$$

the integration to be extended through the given transition-film along the normal n. The value of this integral can be found in precisely the same manner as that of the integral expression for E_r (cf. p. 53); we find then

$$E_f = \frac{\mathfrak{e}}{4\pi}(N_1 - N_0), \quad\text{.................}(22)$$

where

$$N = P\cos(n, x) + Q\cos(n, y) + R\cos(n, z).$$

Differentiating ϕ with regard to n we have

$$\begin{aligned}\frac{d\phi(x, y, z)}{dn} &= \frac{d\phi}{dx}\frac{dx}{dn} + \frac{d\phi}{dy}\frac{dy}{dn} + \frac{d\phi}{dz}\frac{dz}{dn}\\ &= \frac{d\phi}{dx}\cos(n, x) + \frac{d\phi}{dy}\cos(n, y) + \frac{d\phi}{dz}\cos(n, z).\end{aligned}$$

For aphotic motion—formulae (14, V.) then hold—this expression can be written in the form

$$\frac{d\phi}{dn} = -P\cos(n, x) - Q\cos(n, y) - R\cos(n, z).$$

Comparing this expression with that for N, we observe that

$$N = -\frac{d\phi}{dn}.$$

The above expressions for E_r and E_f can therefore be written

$$\left.\begin{aligned}E_r &= \frac{1}{4\pi}\left(D_0\frac{d\phi_0}{dn} - D_1\frac{d\phi_1}{dn}\right)\\ E_f &= \frac{\mathfrak{e}}{4\pi}\left(\frac{d\phi_0}{dn} - \frac{d\phi_1}{dn}\right)\end{aligned}\right\}\quad\text{.............}(23)$$

Since all points on the surface-element do are at approximately the same distance ρ from the point

(x_1, y_1, z_1) at which the potential ϕ is sought, the free electricity residing on this surface will contribute approximately the amount $E_f do/\rho$ to integral (11).

Similarly, the expression for $\frac{dE_r}{dt}$, namely,

$$\frac{dE_r}{dt} = -L_1(N_1+S_1)+L_0(N_0+S_0),$$

(cf. p. 56) can be written as follows for aphotic motion:

$$\frac{dE_r}{dt} = L_1\left(\frac{d\phi_1}{dn} - S_1\right) - L_0\left(\frac{d\phi_0}{dn} - S_0\right), \quad \text{.........(24)}$$

where S denotes the component of the external electromotive forces X, Y, Z along the normal n. When $\phi_1 = \phi_0$, S_1 and S_0 may be rejected and this general expression for $\frac{dE_r}{dt}$ thus reduces to

$$\frac{dE_r}{dt} = L_1\frac{d\phi_1}{dn} - L_0\frac{d\phi_0}{dn}. \quad \text{..................(25)}$$

Still simpler relations than (23) hold for the dividing-surfaces of insulators ($L=0$), provided we only assume that external electromotive forces have never acted in their transition-films, and that initially ($t = -\infty$) no real electricity resided within them; equation (2, III.) ($X = Y = Z = \text{L} = 0$) then gives

$$\frac{d\epsilon_r}{dt} = 0, \text{ hence } \epsilon_r = \text{const.} = 0.$$

It follows therefore that

$$\int \epsilon_r dn = E_r = D_1N_1 - D_0N_0 = 0,$$

or

$$D_1N_1 = D_0N_0, \quad \text{..........................(26)}$$

which assumes the following form for aphotic motion:

$$D_1\frac{d\phi_1}{dn} = D_0\frac{d\phi_0}{dn}. \quad \text{........................(27)}$$

CHAPTER VII.

SECTION XVI. ELECTROSTATICS.

THE advantage gained by the introduction of the conception of the real electricity will become more apparent upon the examination of special problems. We first assume that external electromotive forces have acted at some period of the past, and examine the state of the ether long after their cessation. This state of the ether we shall designate as the electrostatic. As the motion of the ether has surely become aphotic in the meantime, equations (14, V.) will hold. According to formula (7, II.), the amount of energy per unit-volume transformed into Joule's heat during unit-time is now $L(P^2+Q^2+R^2)$. As this transformation of energy into heat cannot continue indefinitely, since the source of energy must otherwise be infinitely large, the expression $L(P^2+Q^2+R^2)$ must finally vanish (cf. also § 18); it is then that we examine the state of the ether. In conductors, where $L \gtrless 0$, P, Q, R will vanish, and ϕ will thus be constant (cf. equations (14, V.)). In bodies for which L vanishes, the insulators, it is not necessary, however, that $(P^2+Q^2+R^2)$ should vanish; on the other hand, we know that for such bodies the real and neutral electricities are bound, and hence should conclude rightly that ϕ can be a function of the coordinates xyz only, and not of the time. The unique existence of ϕ, as well as its constancy with regard to the time, can moreover be deduced from our equations by means of a theorem

from the theory of the potential, which may be stated as follows: If for insulators the volume-density

$$\epsilon_r = -\frac{1}{4\pi}\nabla^2\phi$$

(cf. formula (1, VI.)) and the surface-density

$$E_r = \frac{1}{4\pi}\left(D_0\frac{d\phi_0}{dn} - D_1\frac{d\phi_1}{dn}\right)$$

(cf. formula (22)) are given—these shall be independent of the time—and for every system of conductors the total quantity of electricity on their respective surfaces, namely $\int \frac{do}{4\pi}\frac{Dd\phi}{dn}$, is given, moreover if within all conductors ϕ is constant, and hence $\nabla^2\phi = 0$, and lastly, if ϕ vanishes at an infinite distance, ϕ will then be uniquely determined, being of course entirely independent of the time.* External electromotive forces are of course excluded here.

Equations (14, V.) have been obtained by rejecting certain quantities. Hereby the one great advantage has been gained that we know why these equations hold approximately: they have namely been established by the introduction of changes, which have brought about the rejection of quantities, whose magnitude is small compared to that of the velocity of light. It is evident from the nature of the problem that the right to reject such quantities does not admit of an absolutely rigorous mathematical proof; it will thus be well to have an alternative, to which we may have recourse; if, namely, we put

$$X = Y = Z = 0,$$

and $$P = -\frac{d\psi}{dx},\quad Q = -\frac{d\psi}{dy},\quad R = -\frac{d\psi}{dz},$$

* See Riemann's *Vorlesungen über Schwere, Electricität und Magnetismus*, §§ 18 and 22.

where ψ is an arbitrary function of x, y, z, but is constant with regard to t, and moreover assume that the function ψ is constant in all conductors, equations (9, II.) and (10, II.) will then be satisfied identically (without the rejection of any quantities). For in this case equations (9, II.) reduce to

$$0=\frac{d\beta}{dz}-\frac{d\gamma}{dy}+\frac{4\pi L}{\mathfrak{B}}\frac{d\psi}{dx},$$

$$0=\frac{d\gamma}{dx}-\frac{da}{dz}+\frac{4\pi L}{\mathfrak{B}}\frac{d\psi}{dy},$$

$$0=\frac{da}{dy}-\frac{d\beta}{dx}+\frac{4\pi L}{\mathfrak{B}}\frac{d\psi}{dz},$$

which differentiated, the first with regard to x, the second to y, and the third to z, and added together, give the condition

$$\frac{4\pi L}{\mathfrak{B}}\nabla^2\psi=0.$$

This condition is now always fulfilled, since for insulators L vanishes, and in all conductors ($L\gtrless 0$) ψ has been assumed constant. Under the above assumptions for P, Q, R and ψ, equations (10, II.) give

$$\frac{da}{dt}=\frac{d\beta}{dt}=\frac{d\gamma}{dt}=0, \qquad \text{.....................(1)}$$

that is, a, β, γ are independent of the time, and equations (9, II.) after the elimination of ψ:

$$\left.\begin{aligned}\frac{d^2\gamma}{dxdz}-\frac{d^2a}{dz^2}&=\frac{d^2a}{dy^2}-\frac{d^2\beta}{dxdy}\\ \frac{d^2a}{dxdy}-\frac{d^2\beta}{dx^2}&=\frac{d^2\beta}{dz^2}-\frac{d^2\gamma}{dxdy}\\ \frac{d^2\beta}{dydz}-\frac{d^2\gamma}{dy^2}&=\frac{d^2\gamma}{dx^2}-\frac{d^2a}{dxdz}\end{aligned}\right\}, \qquad \text{..............(2)}$$

from which it follows that

$$\alpha=\frac{df(x, y, z)}{dx}, \quad \beta=\frac{df(x, y, z)}{dy}, \quad \gamma=\frac{df(x, y, z)}{dz} \quad \ldots(3)$$

In the given case we see therefore that α, β, γ are also the derivatives of an arbitrary function f of x, y, z with regard to the coordinates, and that they are likewise independent of the time. $\alpha=\beta=\gamma=0$ would, of course, also satisfy equations (1) and (2).

If we now make the assumption that ψ vanishes at infinite distance, ψ is determined uniquely, for it is then identical to ϕ. We could therefore adopt Kirchhoff's method, which we have already mentioned on page 105, and after we had shown that given values or functions assumed for P, Q, R, $\psi=\phi$, α, β, γ are particular integrals of our general equations, then proceed to interpret their physical meaning.

If we consider ϕ as given, the densities of the free and real electricities at any point of space not only in the interior of all bodies but on their dividing-surfaces will then be determined by equations (1, VI.), (10, VI.), (22, VI.). The problem to be solved is, however, exactly the reverse of this problem; the quantity of real electricity is namely given, and ϕ is to be so chosen (1) that at any point within the given insulators the expression

$$-\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)\right]$$

gives the given density of the real electricity at that point, vanishing of course in those regions, where no real electricity resides, (2) that at any point on the dividing-surface of any two insulators the expression

$$\frac{1}{4\pi}\left(D_0\frac{d\phi_0}{dn}-D_1\frac{d\phi_1}{dn}\right)$$

gives the given surface-density of the real electricity at

that point, vanishing when no surface-density is given and (3) that for every conductor the quantity

$$\int \frac{d\sigma}{4\pi} D \frac{d\phi}{dn},$$

the integration being extended over its whole surface, gives the given amount of real electricity residing on its surface. According to the theorem just cited this problem admits of only one solution (for ϕ).

Let us next examine an insulator, in which D is constant. The density of the real electricity at any point within it will then be given by the expression

$$\epsilon_r = -\frac{D}{4\pi}\nabla^2\phi.$$

Suppose now that the given insulator contains conductors charged with electricity but within which D may assume any value, constant or variable (function of x, y, z), and let us examine this system after ϕ has become constant within the conductors. If do denotes a surface-element of any one of the conductors, the quantity of real electricity that accumulates upon it will be by formula (22, VI.)

$$E_r do = -\frac{D}{4\pi} \cdot \frac{d\phi}{dn}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where D and $\frac{d\phi}{dn}$ refer to the dividing-surface between the transition-film and the insulator and the normal n is drawn from that surface into the insulator. The second term of formula (22, VI.), $+\frac{D}{4\pi} \cdot \frac{d\phi}{dn}$, that which refers to the dividing-surface between the transition-film and the conductor, vanishes, since according to assumption ϕ has already become constant within the conductor, and hence $\frac{d\phi}{dn} = 0$ on the given surface—the normal n is to be drawn here from the film into the conductor.

Similarly, the surface-density of the free electricity on any conductor will be

$$E_f = -\frac{\mathfrak{k}}{4\pi}\cdot\frac{d\phi}{dn}$$

and its volume-density at any point of the insulator

$$\epsilon_f = -\frac{\mathfrak{k}}{4\pi}\nabla^2\phi.$$

It follows therefore that

$$\epsilon_f = \frac{\mathfrak{k}}{D}\epsilon_r, \quad E_f = \frac{\mathfrak{k}}{D}E_r. \quad \text{.....................(5)}$$

For non-homogeneous insulators or dielectrics only the first of these relations (5) will undergo any change; in its place we get

$$\epsilon_f = \frac{\mathfrak{k}}{D}\left[\epsilon_r + \frac{1}{4\pi}\left(\frac{d\phi}{dx}\cdot\frac{dD}{dx} + \frac{d\phi}{dy}\cdot\frac{dD}{dy} + \frac{d\phi}{dz}\cdot\frac{dD}{dz}\right)\right]. \quad \text{...(6)}$$

In this case the expression

$$\frac{\mathfrak{k}}{4\pi D}\left(\frac{d\phi}{dx}\cdot\frac{dD}{dx} + \frac{d\phi}{dy}\cdot\frac{dD}{dy} + \frac{d\phi}{dz}\cdot\frac{dD}{dz}\right)$$

must be regarded as the bound part of the free electricity; by its appearance the real electricity is reduced by the amount

$$\frac{1}{4\pi}\left(\frac{d\phi}{dx}\cdot\frac{dD}{dx} + \frac{d\phi}{dy}\cdot\frac{dD}{dy} + \frac{d\phi}{dz}\cdot\frac{dD}{dz}\right),$$

that is, this amount of real electricity is rendered inactive, due to the heterogeneity of the dielectric.

Let us next determine the ponderable forces of the above system; they follow directly from our fundamental expression for the kinetic energy in Chapter I. Replacing P, Q, R by their values (14, V.) in formula (7, II.), we find the following expression for the kinetic energy of our medium per unit volume:

$$\frac{D}{8\pi}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right];$$

the total electrostatic kinetic energy of our system will thus be

$$\frac{1}{8\pi}\int_U D\left[\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right]d\tau. \quad \ldots\ldots\ldots\ldots(7)$$

Integrating the first term of this expression partially with regard to x, we have

$$\frac{1}{8\pi}\underbrace{\iiint}_{U} D\frac{d\phi}{dx}d\phi\, dy\, dz$$

$$=\frac{1}{8\pi}\iint\left|\phi D\frac{d\phi}{dx}\right|dy\,dz-\frac{1}{8\pi}\int_U\phi\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)d\tau.$$

The surface-integral of this equation vanishes in conformity to our principle of the continuity of transitions, and we have

$$\frac{1}{8\pi}\int_U D\left(\frac{d\phi}{dx}\right)^2 d\tau=-\frac{1}{8\pi}\int_U\phi\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)d\tau,$$

and similarly analogous transformations for the other two integrals of expression (7); the given expression can thus be written

$$-\frac{1}{8\pi}\int_U\phi\left[\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)\right]d\tau,$$

or by formula (1, VI.) $\frac{1}{2}\int_U\phi\epsilon_r d\tau.$

Lastly, replacing here ϕ by its value (11, VI.) we get

$$\frac{1}{2\varepsilon}\int_U\int_U\frac{\epsilon_r\epsilon_f' d\tau\, d\tau'}{\rho}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where $d\tau'$ denotes any element of the ether, and ϵ_f' the density of the free electricity within that element; the index U denotes here, as above (p. 116), that the

dividing-surfaces of the given media are to be included in the integrations. Explicitly we should write expression (8) in the form

$$\frac{1}{2\epsilon}\underbrace{\iiint}_{U}\underbrace{\iiint}_{U}\frac{\epsilon_r(xyz)\epsilon_r(x'y'z')dx\,dy\,dz\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}.$$

This is now the expression for the total electrostatic kinetic energy of the ether; it is also its total *aphotic* kinetic energy. For D constant this expression reduces by formula (6) to

$$\frac{1}{2D}\underbrace{\iiint}_{U}\underbrace{\iiint}_{U}\frac{\epsilon_r(xyz)\epsilon_r(x'y'z')dx\,dy\,dz\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}; \quad \ldots.(9)$$

for D variable its explicit form would be

$$\frac{1}{2D}\underbrace{\iiint}_{U}\underbrace{\iiint}_{U}\frac{\epsilon_r(xyz)\epsilon_r(x'y'z')dx\,dy\,dz\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$-\frac{1}{4\pi D}\underbrace{\iiint}_{U}\underbrace{\iiint}_{U}\frac{\epsilon_r(xyz)\left[\frac{d\phi(x'y'z')}{dx'}\cdot\frac{dD(x'y'z')}{dx'}+\frac{d\phi(x'y'z')}{dy'}\cdot\frac{dD(x'y'z')}{dy'}+\frac{d\phi(x'y'z')}{dz'}\cdot\frac{dD(x'y'z')}{dz'}\right]dx\,dy\,dz\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}},$$

which is of no practical value, since ϕ must be known as a function of the coordinates before the second integral can be evaluated. We should, however, observe that this second integral may be rejected provided D is variable in the conductors only.

Let us next divide the total quantity of electricity in our system into two groups, suffixing the index 1 to the volume-elements within which electricity of the first group resides, and to the densities within those elements, and the index 2 to the volume-elements and densities

of the second group; let two charged conductors, within whose volume-elements $d\tau_1$ and $d\tau_2$ the density of the real electricity is ϵ_1 and ϵ_2 respectively, placed in a dielectric (air) at a given distance ρ apart, serve as an illustration. The expression for the total electrostatic energy can then be written:

$$\frac{1}{D}\left[\frac{1}{2}\iint\limits_{U\,U}\frac{d\tau_1 d\tau_1'\epsilon_1\epsilon_1'}{\rho}+\frac{1}{2}\iint\limits_{U\,U}\frac{d\tau_2 d\tau_2'\epsilon_2\epsilon_2'}{\rho}+\iint\limits_{U\,U}\frac{d\tau_1 d\tau_2\epsilon_1\epsilon_2}{\rho}\right], \ldots(10)$$

where, for simplicity, as only real electricities are concerned, the suffix r has been dropped. We shall call the first integral of this expression, which is the potential of the electricities of the first group on themselves, the self-potential of the electricities of the first group, and the second integral the self-potential of the electricities of the second group; whereas the third integral, the potential of the electricities of the one group on those of the other, shall be called the mutual potential of the two groups, and be denoted by T_{12}.

Suppose now that the electricities of the first group retain their same relative positions, and those of the second group theirs, but that the relative position of the former to the latter changes; in the illustration just used this could be effected by a slight variation in the distance ρ between the two conductors. In this case only the third integral T_{12} of the above expression (10) would undergo any change, and this change would be due to the variation in the distance ρ between the volume-elements $d\tau_1$ and $d\tau_2$; ρ would undergo no variation in the other two integrals. Since we have assumed that no energy except that transformed into Joule's heat is spent, an amount of energy exactly equal to this variation in T_{12} must appear in the form of visible kinetic energy or ponderable forces doing work. The expression

$$-\delta T_{12}=\frac{1}{D}\delta\iint\limits_{U\,U}\frac{d\tau_1 d\tau_2\epsilon_1\epsilon_2}{\rho}, \ldots\ldots\ldots\ldots\ldots(11)$$

must then be the work done by the forces which act apparently at a distance. We obtain now exactly this expression for the work done from our concrete representation, by assuming that every quantity of real electricity e_r repels every other quantity e_r' with a force equal to

$$\frac{e_r e_r'}{D\rho^2}. \quad \text{.........................(12)}$$

Our medium contains, however, not only the real electricity, but that due to its electric polarization; consequently we must assume that in our concrete representation every type of electricity, the sum of which is the total free electricity, acts on every quantity of real electricity e_r in the medium and, moreover, in order to agree with Maxwell's theory that every quantity of free electricity e_f' repels every quantity of real electricity e_r, according to the law

$$\frac{e_r e_f'}{\mathfrak{k}\rho^2}. \quad \text{.........................(13)}$$

The laws of repulsion (12) and (13) can easily be shown to be consistent with each other (cp. formula (5)). It thus follows that we can illustrate the ponderable forces in our concrete representation by assuming that the free electricity acts either on the neutral electricity or on the real electricity, and in both cases according to the same law.

Since the free electricity at any point is the sum of all the active electricities, to be consistent we must, moreover, assume in our concrete representation that every quantity of free electricity in the medium acts on every other quantity of free electricity according to the above law, namely

$$\frac{e_f e_f'}{\mathfrak{k}\rho^2}; \quad \text{.........................(14)}$$

the ponderable forces will not, however, follow directly

from this law (cf. next article). One reason why we decided above (cf. p. 47) in favour of the dualistic theory for our concrete representation is that the action of the neutral electricity not displaced can always be excluded, since, according to this theory, the neutral electricity not displaced is composed of the same quantities of positive and negative fluid; whereas, if we had accepted the unitary theory, we should now be obliged to assume the action of ponderable matter (cf. Maxwell's treatise, § 37).

Let us next choose some real medium, for example, the atmosphere, as standard, putting $D=1$. In this medium the observed repulsion of two quantities of real electricity e_r and e_r' would then be according to law (12)

$$\frac{e_r e_r'}{\rho^2}.$$

We can thus define the electrostatic unit of force as the force with which two unit-quantities of electricity repel each other in the standard medium at unit-distance. P, Q, R are now the components of the resultant force that acts on unit-quantity of electricity; measured in the electrostatic system of units employed in § 4, their dimensions are

$$m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}.$$

Since $$\epsilon = \frac{1}{4\pi}\left[\frac{d(DP)}{dx} + \frac{d(DQ)}{dy} + \frac{d(DR)}{dz}\right],$$

its dimensions in this electrostatic system will be

$$[\epsilon] = m^{\frac{1}{2}} l^{-\frac{3}{2}} t^{-1}$$

and hence those of any quantity of real electricity e_r

$$[e_r] = [\epsilon] l^3 = m^{\frac{1}{2}} l^{\frac{3}{2}} t^{-1}.$$

The mutual repulsion of two quantities of real electricity

e_r and e_r', namely $\frac{e_r e_r'}{\rho^2}$, measured in the electrostatic system, will therefore have the dimensions

$$mlt^{-2},$$

that is, those of a force.

Let us denote the value of $D \lesseqgtr 1$ for any other medium than the standard medium by D; the observed repulsion (attraction) of two quantities of real electricity e_r and e_r' will then be

$$\frac{e_r e_r'}{D\rho^2},$$

(cf. formula (12)), that is, it will be equal to the mutual repulsion of these two quantities of electricity e_r and e_r' in the standard dielectric (air) divided by D, where D is the ratio of the constants of the electric polarization of the two dielectrics. Hence two quantities of electricity, $e_r = e_r' = \sqrt{D}$, measured in the standard dielectric, would repel each other with unit-force when placed in the new medium $D \gtreqless 1$; $\sqrt{D}$ must therefore be chosen as unit-quantity of electricity in the new dielectric. Denoting the number of units of electricity in any quantity of electricity measured in the old medium and in the new standard medium by ϵ and ϵ_h respectively, we have then the relation

$$\epsilon = \epsilon_h \sqrt{D}. \qquad (15)$$

We can thus always obtain the equations that hold for the electrostatic system of units of the new standard medium by putting $h = \sqrt{D}$ in the formulae at the end of § 8.

SECTION XVII. ASSUMPTION THAT $\mathfrak{e}$ IS SMALL IN COMPARISON TO UNITY. THE ACTION AT A DISTANCE OF THE ELECTRICITY DUE TO ELECTRIC POLARIZATION (INDUCTION).

Our concrete representation assumes the simplest form when we put $\mathfrak{e}=1$ in our standard medium, $D=1$ (air), that is, when we assume that our standard body cannot be polarized (cf. formula (16, VI.)). The quantity ϵ_{vH} of equation (17, VI.) then vanishes for the standard medium, and the inductive capacity D of any other body thus becomes

$$D=1+4\pi\epsilon_{vH}. \quad (\mathfrak{e}=1).$$

It is possible that in certain media, as in hydrogen or a vacuum, $D<1$, and hence that ϵ_{vH} becomes negative; the electric polarization would then be of the opposite kind, and we should be obliged to form the same conception of it as we are often wont to form of the magnetic polarization or induction. Partly to avoid this, but chiefly for other reasons to be mentioned directly, von Helmholtz conceived the idea of assigning $\mathfrak{e}$ a value $\mathfrak{e}_i$, that is small in comparison to unity. Let us next imagine a dielectric for which $D_i=\mathfrak{e}_i$; as it is not essential that such a dielectric should exist, we shall designate it as our *ideal* standard medium. Unit-quantity of real electricity in the electrostatic system of units of our real standard medium would thus repel in our ideal standard medium, where $\epsilon_p=0$ (cf. formula (16, VI.)), the same unit-quantity of electricity at unit-distance, not with unit-force, but with the force $1/\mathfrak{e}_i$; this follows from equation (13). Since ϵ_p now vanishes, ϵ_f may be put equal to ϵ_r and formula (15) may thus be written

$$\epsilon_r{}^i=\frac{\epsilon_r}{\sqrt{\mathfrak{e}_i}},$$

where ϵ^i_r is the quantity of real electricity measured in the electrostatic system of units of the ideal standard medium. The transformation of our equations to the electrostatic system of units of our ideal standard medium can be effected directly by replacing h by $\sqrt{\mathfrak{k}_i}$ in the equations at the end of § 8. In the ideal medium ϵ_f would now be equal to ϵ_r, since $\epsilon_p=0$; and in the real medium

$$\epsilon_f=\mathfrak{k}_i\epsilon_r$$

according to formula (5, VII.), that is, ϵ_f would be small in comparison to ϵ_r. Hence, since $\epsilon_f=\epsilon_r+\epsilon_p$, ϵ_p and $-\epsilon_r$ would be almost equal to each other. We could thus designate $1/\mathfrak{k}_i$ as the inductive capacity of the real standard medium referred to the ideal standard medium, the latter being taken as unit.

If we put $\mathfrak{k}=1$ in the real standard medium, for which $D=1$, we then exclude the possibility of its electric polarization. Since now two quantities of real electricity e_r and e_r' apparently act on each other at a distance according to the law expressed by equation (12), the potential of all the electricities of the medium will then be that of all its real electricity; this potential measured in the electrostatic system of units of this real standard medium is evidently

$$\tfrac{1}{2}\iint\frac{\epsilon_r\epsilon_r'd\tau\, d\tau'}{\rho},$$

which by formula (8) is the expression for the total electrostatic energy of the medium. It thus follows that the total electrostatic energy of the given medium is equal to the potential due to the forces that act apparently at a distance between its real electricities. Hence, when the distance between two quantities of electricity, as two charged conductors, increases and thus either work is done or visible kinetic energy appears, Maxwell supposes that the energy of the medium is transformed into work or visible kinetic energy; whereas

we conceive (in our concrete representation) that the potential of the real electricity of the medium, that is, that of all its electricities—the real electricity is here the only electricity present, since ϵ_p vanishes, $\mathfrak{e}$ having been put equal to unity—diminishes by exactly the same amount as its visible kinetic energy increases.

If we assign $\mathfrak{e}$ any other value than unity in our real standard medium, $D=1$, we must then evidently distinguish between the real and free electricities, for the potential of all the electricities in the medium is then no longer that of only the real electricity, but it also includes that due to its electric polarization (cf. p. 128).

Retaining the above electrostatic system of units, we have the following expression for the potential of the free electricities in our standard medium, $D=1$, $\mathfrak{e} \lesseqgtr 1$:

$$\tfrac{1}{2}\iint\frac{\epsilon_f\epsilon_f'}{\rho}d\tau\,d\tau' = \frac{\mathfrak{e}}{2}\int\phi\epsilon_r d\tau \quad\text{................(16)}$$

(cf. formula (11, VI.)), which is $\mathfrak{e}$ times the electrostatic kinetic energy of the system. If now a given quantity K of this is transformed into visible kinetic energy, the potential of all the electricities will be diminished only by $\mathfrak{e}K$—we suppose $\mathfrak{e}<1$,—and there will remain the work

$$(1-\mathfrak{e})K$$

done, but not accounted for. This work is to be attributed to the molecular forces that resist the electric polarization (in insulators), for the expression for the work done by the forces of electric polarization, namely,

$$\int(\mathfrak{x}^2+\mathfrak{y}^2+\mathfrak{z}^2)d\tau,$$

decreases for positive K. According to formula (18, VI.) the total work done by these forces is

$$\int\frac{1-\mathfrak{e}}{8\pi}(P^2+Q^2+R^2)d\tau=(1-\mathfrak{e})\int Td\tau.$$

Maxwell conceives this energy as a potential energy stored up in the insulator, and appearing when its electric polarization begins. As ϵ approaches unity, we have the limiting case, where all the visible kinetic energy is derived from the molecular forces of the insulator, and none from the potential of its free electricity. We should not fail, however, to observe that by putting $\epsilon=0$ we exclude the existence of all free electricity.

As ϵ decreases, the density of the free electricity

$$\frac{\epsilon}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)$$

diminishes, and the absolute value of the density of the real electricity thus approaches the absolute value of that of the electricity due to electric polarization. If we write equation (2, III.) in the form

$$-\frac{d\tau}{4\pi}\frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]$$
$$=d\tau\left[\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}\right]$$

and assume that ϵ is infinitely small, so that any expression containing it as factor may be rejected, it follows then from formula (16, VI.) that the left-hand member of this equation is the increment of the electricity in the volume-element $d\tau$ due to the variation in its electric polarization, whereas its right-hand member evidently represents the diminution of the real electricity in this element, that is, the electricity withdrawn from it by conduction. Their sum is zero, that is, the quantity of active electricity withdrawn from every element $d\tau$ by conduction is the same as that created within it by electric polarization or induction. Here the creation of real electricity in certain regions is thus due to the flow (conduction) of the neutral electricity of the first type, the so-called current electricity mentioned on

page 113; the same quantity of neutral electricity of the second type must therefore be withdrawn from these regions by electric polarization. It follows of course that an equal amount of neutral electricity of the first and second types must be withdrawn by conduction from and introduced by electric polarization respectively into other regions of the medium. We see therefore that, if we assume that ϵ is very small, the first and second types of neutral electricity, that is, the current electricity and that due to electric polarization, will behave like two approximately incompressible fluids. Poincaré lays great stress on this analogy, employing it on all possible occasions. It would, however, be just as wrong to suppose that Maxwell believed in the existence of two such fluids, as to think that he conceived lines of force to be real threads or tubes pervading space; in fact, he makes use of quite a different analogy or dynamical illustration,* which has been developed by Gordon. The latter conceives a dielectric or insulator as consisting of a system of spheres or beads strung on non-elastic strings and the resisting forces (friction) called into action by the displacement of these beads along the strings—they tend to bring the beads back to their initial positions—as representing the molecular forces of the medium; these resisting forces are assumed to be entirely wanting in conductors. The unelectrified state of the medium is supposed to be represented by a uniform distribution of the beads along the strings; wherever they approach one another, that is, in those regions where there is an accumulation of beads, positive electricity is supposed to reside, and wherever they recede from one another, that is, in regions where there is a diminution in their number, negative electricity.

We shall see later that, when ϵ is assumed to be small in comparison to unity, our concrete representation will

* *A dynamical theory of the electro-magnetic field*, Scientific Papers, I., p. 526. *Transactions of the Royal Society*, v. CLV., p. 459, 1864.

both answer for the Hertzian oscillations and be found to agree with the several consequences deduced from the equations of Chapter I.; this will not be true, however, for other values of $\mathfrak{k}$. Quantitative measurements of these phenomena would therefore offer a means of determining the value of this constant, or at least of confirming the supposition that $\mathfrak{k}$ must be taken very small in order that the phenomena of electrodynamics may also follow from our concrete representation. The agreement between the constant of electric polarization determined electrostatically and its value derived from the Hertzian oscillations, can moreover be regarded as a kind of proof of this assumption. Compare also § 37.

The above considerations hold only for a given homogeneous dielectric. If the medium is composed of several dielectrics, each characterized by its constant D of electric polarization, or, if D is a function of x, y, z in any given dielectric, we should have in addition to the above forces those due to the variation in D from point to point (cf. formula (16, VI.)). Very little is known about these forces; Maxwell has shown that the corresponding magnetic forces manifest themselves by tending to move bodies of greater M (D) towards regions of greater field-strength. In our concrete representation, these are evidently the forces exerted by the free electricity on that part of the electricity due to electric polarization arising from any variation in D (cf. formula (16, VI.) and law (13)).

Up to this point we have followed von Helmholtz in assuming that the force that acts on any body is equal to that which acts on its real electricity, without any reference to the electric polarization of the surrounding medium (cf. law (13)). Von Helmholtz explains this as follows: "Für die Verschiebungen von E im Raume S, soweit ϵ constant ist, bildet diese neutralisirende Electricität kein Hinderniss, weil sie überall mitfolgen kann. Die Anziehungskräfte also, welche von anderweitig vorhandenen electrischen Massen

auf E ausgeübt werden, müssen ebenso gross sein, als wenn die E zum Theil neutralisirende Electricität gar nicht vorhanden wäre."* To illustrate the evident meaning of this statement, let us examine two charged conductors E and E' placed in a dielectric; there are then four quantities of electricity whose action comes into consideration, namely, the actions of

(1) E_r', the real electricity in E', on E_r, the real electricity in E.

(2) E_p', the electricity on the surface of E' due to the electric polarization of the surrounding dielectric, on E_r.

(3) E_r' on E_p, the electricity on the surface of E due to electric polarization.

(4) E_p' on E_p.

Von Helmholtz now assumes that only the actions (1) and (2) have any effect on the displacement of the conductor E in space S, as in a fluid; he attempts to explain this by the fact that the electricity due to electric polarization does not reside in the conductor, but in the medium. Such an explanation seems to me to be somewhat forced even in the most ideal case, where the dielectric is a vacuum or air; whereas for such dielectrics as oil it is evidently quite unsatisfactory, since a part of the polarized oil will surely cling to the surface of the conductor, and thus be acted upon exactly as if it belonged to or resided within it. Such an assumption or explanation could of course form a new feature of our concrete representation, but to regard it as anything material or really explanatory seems to me to be forcing the point. The external action on the neutralizing electricity E_p might indeed produce a stress in the surrounding medium or fluid; an electrically polarized body immersed in a (polarized) fluid is in fact subjected to such an indirect action. On the other hand, it follows from the principle

* Von Helmholtz's *Wissenschaftliche Abhandlungen*, v. I., p. 614.

of the conservation of energy that, although these forces would evidently have to be taken into consideration in the theory of direct action at a distance, they play no rôle here. It is, however, impossible to get a clear insight into this subject before we have taken up the theory of so-called electro and magneto-striction (cf. § 41). As the forces (stress) due to electro-striction have not yet been derived from the mechanical properties of the medium, they have been found from the principle of the conservation of energy, the latter must be regarded as the real solution of all our present deductions.

If the body within which the two types of electricity reside were a solid, it would only be possible to observe the action of the forces tending to move any one of its volume-elements by boring a small hole round it and filling it with a fluid; and if the constant of electric polarization of the fluid were not the same as that of the given body, the hole would have to be bored in the form of a cylinder, whose dimensions in the direction of the resultant electric forces were large in comparison to those of its cross-section (cf. § 26).

CHAPTER VIII.

SECTION XVIII. STATE OF THE ETHER PREVIOUS TO THE ELECTROSTATIC STATE AND TRANSITION TO THE LATTER, $X=Y=Z=0$.

Let us suppose that an electric disturbance is started at any point O of the ether—we can choose this point as origin of our system of coordinates. If we assume that the ether is constituted symmetrically with regard to the point O, that is, that all quantities are functions only of r, the radius vector of any point A from O, and the time t, the disturbance will be propagated radially into space. Let us then examine the state of the medium directly after the passage of the electro-magnetic waves, that is, its aphotic state. We know now that this state is characterized by the existence of a potential ϕ for the electromotive forces P, Q, R, namely,

$$P=-\frac{d\phi}{dx},\quad Q=-\frac{d\phi}{dy},\quad R=-\frac{d\phi}{dz}. \qquad\text{............(1)}$$

Since all quantities are symmetrical with regard to O, ϕ can be regarded as a function of r and t only instead of x, y, z and t, and P, Q, R can thus be written,

$$P=-\phi'\frac{x}{r},\quad Q=-\phi'\frac{y}{r},\quad R=-\phi'\frac{z}{r}, \qquad\text{..........(2)}$$

where $$\phi'=\frac{d\phi}{dr},$$

from which it follows that the resultant electromotive force along the vector r—we denote it by S—is

$$\sqrt{P^2+Q^2+R^2}=\phi'=S,$$

the density of the real electricity at any point A, namely,

$$\epsilon_r=+\frac{1}{4\pi}\left[\frac{d}{dx}(DP)+\frac{d}{dy}(DQ)+\frac{d}{dz}(DR)\right],$$

can then be written

$$\epsilon_r=-\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\phi'\frac{x}{r}\right)+\frac{d}{dy}\left(D\phi'\frac{y}{r}\right)+\frac{d}{dz}\left(D\phi'\frac{z}{r}\right)\right];$$

or, since

$$\frac{d}{dx}\left(D\frac{\phi'}{r}\right)=\frac{d}{dr}\left(D\frac{\phi'}{r}\right)\frac{dr}{dx}=\frac{d}{dr}\left(D\frac{\phi'}{r}\right)\frac{x}{r},$$

and similarly

$$\frac{d}{dy}\left(D\frac{\phi'}{r}\right)=\frac{d}{dr}\left(D\frac{\phi'}{r}\right)\frac{y}{r},\quad \frac{d}{dz}\left(D\frac{\phi'}{r}\right)=\frac{d}{dr}\left(D\frac{\phi'}{r}\right)\frac{z}{r},$$

$$\epsilon_r=-\frac{1}{4\pi}\left[\frac{3D\phi'}{r}+\frac{d}{dr}\left(\frac{D\phi'}{r}\right)\cdot\frac{x^2+y^2+z^2}{r}\right]$$

$$=-\frac{1}{4\pi}\left[\frac{3D\phi'}{r}+r\frac{d}{dr}\left(\frac{D\phi'}{r}\right)\right].\quad\ldots\ldots\ldots\ldots(3)$$

Similarly equation (2, III.) for $\frac{d\epsilon_r}{dt}$ reduces by formula (2) to

$$\frac{d\epsilon_r}{dt}=\frac{3L\phi'}{r}+r\frac{d}{dr}\left(\frac{L\phi'}{r}\right).\quad\ldots\ldots\ldots\ldots(4)$$

This equation could of course also be obtained from our concrete representation by considering the flow of the electric fluids.

If we now assume that the medium is homogeneous, D and L constant, we can write equations (3) and (4) as follows:

$$\epsilon_r = -\frac{D}{4\pi}\left[\frac{3\phi'}{r} + r\frac{d}{dr}\left(\frac{\phi'}{r}\right)\right] = -\frac{D}{4\pi}\left[\frac{2\phi'}{r} + \phi''\right]$$

and $\quad \dfrac{d\epsilon_r}{dt} = L\left[\dfrac{2\phi'}{r} + \phi''\right]$, where $\phi'' = \dfrac{d^2\phi}{dr^2}$.

These equations give

$$\frac{d\epsilon_r}{dt} = -\frac{4\pi L}{D}\epsilon_r,$$

the integral of which is

$$\epsilon_r = Ce^{-\frac{4\pi L}{D}t}.$$

From the condition that for $t = t_0$, the beginning of the aphotic motion, the density of the real electricity ϵ_r at any point is ϵ_r^0, we find the value ϵ_r^0 for our arbitrary constant C, and we thus have

$$\epsilon_r = \epsilon_r^0 e^{-\frac{4\pi L}{D}t}. \qquad (5)$$

For good conductors—such as the metals—L is very large, and hence the exponent of e very small; consequently after a very short period (the millionth part of a second for metals) the factor $e^{-\frac{4\pi L}{D}t}$ for good conductors will approximately vanish, and hence ϵ_r also.

Since D and L are constant, we have by formula (1, VI.)

$$\epsilon_r = -\frac{D}{4\pi}\nabla^2\phi,$$

and by formula (10, VI.)

$$\epsilon_f = -\frac{\mathfrak{k}}{4\pi}\nabla^2\phi,$$

which give

$$\epsilon_f = \frac{\mathfrak{k}}{D}\epsilon_r, \text{ hence } \epsilon_f^0 = \frac{\mathfrak{k}}{D}\epsilon_r^0$$

(cf. also formulae (5, VII.)), or by the above value (5) for ϵ_r,

$$\epsilon_f = \frac{\mathfrak{k}}{D}\epsilon_r{}^0 e^{-\frac{4\pi L}{D}t} = \epsilon_f{}^0 e^{-\frac{4\pi L}{D}t}.$$

The potential ϕ at any point of the medium will therefore be by formula (11, VI.)

$$\phi = \frac{1}{\mathfrak{k}}\int\frac{\epsilon_f d\tau}{\rho} = \frac{1}{\mathfrak{k}}\int\frac{\epsilon_f{}^0 e^{-\frac{4\pi L}{D}t}d\tau}{\rho} = \frac{e^{-\frac{4\pi L}{D}t}}{\mathfrak{k}}\int\frac{\epsilon_f{}^0 d\tau}{\rho} = \frac{e^{-\frac{4\pi L}{D}t}}{\mathfrak{k}}\phi^0. \quad \ldots(6)$$

We see therefore that ϕ decreases as t increases, and at the same rate as ϵ_f and ϵ_r decrease. It is this vanishing of ϕ or ϵ_f that characterizes the electrostatic state of the ether (cf. the preceding chapter). For very large values of L it is evident that the electrostatic and aphotic states of the ether will set in approximately together.

In the above investigations the assumption that ϕ is a function of r only was superfluous; this becomes evident upon differentiating our fundamental equations (9, II.), the first with regard to x, the second to y, and the third to z, and adding. Assuming that $X = Y = Z = 0$, and that D and L are constant, we find then

$$D\frac{d}{dt}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right) = -4\pi L\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right),$$

which by formula (1, VI.) can be written as follows:

$$\frac{d\epsilon_r}{dt} = -\frac{4\pi L}{D}\epsilon_r,$$

as above. We see therefore that the laws (5) and (6) just found for the behaviour of ϵ_f and ϕ are quite general.

In the above cases we have assumed a homogeneous medium pervading all space. Such a system is of course only theoretically possible. Let us next examine a homogeneous medium of finite dimensions, that is, a medium within which D and L are constant, but at

whose boundaries these quantities which characterize the medium become discontinuous. Instead of investigating here the general case, let us consider as above the special one, where, namely, the electric disturbance is propagated (radially) from the centre of a homogeneous (metallic) sphere.

It is evident that the expressions for ϵ_r, ϵ_f, ϕ, etc., at any point within the sphere will be exactly the same as in the preceding case, where the medium pervaded all space. For the surface of the sphere we shall have, however, the conditions

$$E_r = \frac{1}{4\pi}\left(D_0\frac{d\phi_0}{dr} - D_1\frac{d\phi_1}{dr}\right),$$

$$\frac{dE_r}{dt} = L_1\frac{d\phi_1}{dr} - L_0\frac{d\phi_0}{dr}$$

(cf. formulae (23, VI.) and (25, VI.)), where the indices 0 and 1 are to be interpreted as indicated in the annexed

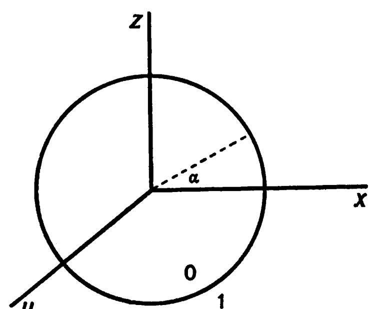

FIG. 10.

figure. The expressions for ϵ_r, ϵ_f, etc., at any point outside the sphere can of course be found by suffixing the index 1 instead of the index 0 to the respective quantities in the expressions already given for the sphere.

Let us next test the above formulae for the following special case. We suppose that the medium which

surrounds the sphere is a non-conductor, air, where $L_0=0$, and determine the amount of real electricity that accumulates on its surface from the beginning of the aphotic motion, $t=0$, to the total subsidence of the ether, $t=\infty$. Since the real electricity, to use our descriptive representation, cannot leave the surface of the sphere and pass off into the insulator ($L_1=0$), this amount will be exactly equal to the quantity of real electricity contained within the sphere at the beginning of the disturbance; this law is known as the conservation of the surface and volume-electricities; to prove it we determine these two quantities.

The expression for the variation in the surface-density of the real electricity E_r is here

$$\frac{dE_r}{dt}=-L_0\frac{d\phi_0}{dr}.$$

Integrating this expression from $t=0$ to $t=\infty$, we find the surface-density of the real electricity that finally passes from the interior of the conductor to the given point of its surface; if we denote its density by E_r^∞, we have

$$E_r^\infty=\int_0^\infty\frac{dE_r}{dt}dt=-L_0\int_0^\infty\frac{d\phi_0}{dr}dt.$$

Since E_r^∞ will have the same value at every point on the surface of the sphere, for all quantities, including E_r^∞, are functions of r and t only, the total quantity of real electricity that finally accumulates on its surface will be

$$4\pi a^2E_r^\infty=-4\pi a^2L_0\int_0^\infty\frac{d\phi_0}{dr}dt, \text{..............}(7)$$

where a is the radius of the sphere.

We have seen on p. 142 that

$$\phi=e^{-\frac{4\pi L}{D}t}\phi^0. \quad (\epsilon=1.)$$

Replacing ϕ by this value in equation (7), we get

$$4\pi a^2 E_r^{\infty} = -4\pi a^2 L_0 \frac{d\phi_0^0}{dr}\int_0^{\infty} e^{-\frac{4\pi L}{D_0}t}dt,$$

which gives

$$4\pi a^2 E_r^{\infty} = -a^2 D_0 \frac{d\phi_0^0}{dr}. \qquad (8)$$

If we denote the density of the real electricity at the beginning of the disturbance, $t=0$, by ϵ_r^0, the quantity of electricity contained in the spherical shell, whose surfaces are generated by the vectors r and $r+dr$, at the time $t=0$ will be

$$4\pi\epsilon_r^0 r^2 dr.$$

The initial quantity of real electricity in the sphere will therefore be

$$4\pi\int_0^a \epsilon_r^0 r^2 dr. \qquad (9)$$

By formula (1, III.) we have

$$\epsilon_r^0 = -\frac{D_0}{4\pi}\nabla^2\phi_0^0 = -\frac{D_0}{4\pi}\left(\frac{d^2\phi_0^0}{dr^2}+\frac{2}{r}\frac{d\phi_0^0}{dr}\right),$$

and expression (9) can thus be written,

$$4\pi\int_0^a \epsilon_r^0 r^2 dr = -D_0\int_0^a\left(r^2\frac{d^2\phi_0^0}{dr^2}+2r\frac{d\phi_0^0}{dr}\right)dr = -D_0\int_0^a d\left(r^2\frac{d\phi_0^0}{dr}\right),$$

which, integrated through the sphere, gives

$$4\pi\int_0^a \epsilon_r^0 r^2 dr = -a^2 D_0\frac{d\phi_0^0}{dr}. \qquad (10)$$

We see therefore that the quantity of real electricity that finally accumulates on the surface of the sphere will be exactly equal to the total quantity of real electricity in the sphere at the beginning of the disturbance. This empirical law, the conservation of the surface and volume-electricities, also holds for any insulated conductor.

SECTION XIX. STATE OF THE ETHER PREVIOUS TO THE STATIONARY STATE AND TRANSITION TO THE LATTER, $X=\frac{d\psi}{dx}$, $Y=\frac{d\psi}{dy}$, $Z=\frac{d\psi}{dz}$. BEHAVIOUR OF THE REGIONS WHERE EXTERNAL ELECTROMOTIVE FORCES RESIDE.

In the preceding article external electromotive forces had already ceased to act before the period under consideration had set in. Let us next examine the state of the medium in given systems during the action of external electromotive forces, that have a potential, namely,

$$X=\frac{d\psi}{dx},\quad Y=\frac{d\psi}{dy},\quad Z=\frac{d\psi}{dz},\ \ldots\ldots\ldots\ldots\ldots(11)$$

but that remain constant with regard to the time t.

It is very doubtful whether electromotive forces can be generated in insulators. Certain crystals, as turmalin and topaz, become electrified when heated (pyro-electricity) or subjected to pressure (piezo-electricity); in all probability the forces produced by these electrifications are, however, confined to the surfaces of the crystals. With this possible exception all electromotive forces are limited to the surface of insulators. In addition to the forces arising from pyro- and piezo-electricities, there are the electromotive forces arising from friction. The phenomena that occur during the creation of these electricities are, however, completely obscured to us, whereas those that succeed them are of no importance. In the following we shall, therefore, exclude the action of all external electromotive forces in insulators, and put

$$X=Y=Z=0.$$

Let any system of conductors and insulators be given and the external electromotive forces (11) reside within certain regions of the former, and let us examine the

aphotic state of the ether, that is, its state directly after the passage of the electromagnetic waves, that are produced on the appearance of these forces and pass off into space with the velocity of light. These slow disturbances of the ether are now characterized by the existence of a potential ϕ for the electromotive forces P, Q, R, namely,

$$P=-\frac{d\phi}{dx}, \quad Q=-\frac{d\phi}{dy}, \quad R=-\frac{d\phi}{dz}.$$

Introducing these values for P, Q, R in our fundamental equations of action (9, II.) and (10, II.) we have

$$\frac{M}{\mathfrak{V}}\frac{d\alpha}{dt}=0, \quad \frac{M}{\mathfrak{V}}\frac{d\beta}{dt}=0, \quad \frac{M}{\mathfrak{V}}\frac{d\gamma}{dt}=0,$$

$$\left.\begin{array}{ll} \text{hence} & \alpha=\text{const. }(t),\ \beta=\text{const. }(t),\ \gamma=\text{const. }(t), \\ \text{and} & \dfrac{D}{\mathfrak{V}}\dfrac{d^2\phi}{dt\,dx}+\dfrac{d\beta}{dz}-\dfrac{d\gamma}{dy}+\dfrac{4\pi L}{\mathfrak{V}}\left(\dfrac{d\phi}{dx}-\dfrac{d\psi}{dx}\right)=0, \end{array}\right\} \quad \ldots(12)$$

with two similar equations in y and z. Before proceeding to the discussion of these general equations let us employ them to examine the following special system:

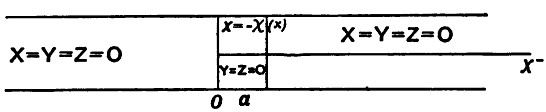

FIG. 11.

a very long cylindrical conductor of cross-section Q is placed in an insulator, air, and external electromotive forces are brought to act in a given cross-section of it; the axis of the cylinder is taken as x-axis and the given forces have the form $Y=Z=0$, $X=-\chi(x)$, that is, the resultant external electromotive force acts along the x-axis, and is a function of x only; lastly, the breadth of the given cross-section is denoted by a, and its left end is chosen as yz-coordinate-plane (cf. figure 11). This case could be realized by inserting a galvanic element

in the given cross-section, or by maintaining a difference of temperature at its two ends. It is evident that all quantities would then be approximately functions of x and t only and not of y and z. We next seek the particular solution of the above equations (12), that corresponds to this special case. Under these limitations and assumptions equations (12) reduce to

$$\alpha = f_1(x), \quad \beta = f_2(x), \quad \gamma = f_3(x),$$

and hence $$D\frac{d^2\phi}{dt\,dx} + 4\pi L\frac{d\dot{\phi}}{dx} + 4\pi L\chi(x) = 0,$$

or $$D\frac{dP}{dt} + 4\pi LP = 4\pi L\chi(x)\,; \quad\text{..............(13)}$$

Q and R vanish. The integral of equation (13) has been found by Euler and expressed in the general form

$$P = e^{-\int\frac{4\pi L}{D}dt}\int\frac{4\pi L}{D}\chi(x)e^{+\int\frac{4\pi L}{D}dt}dt. \quad\text{.............(14)}$$

These integrals cannot of course be evaluated until given values are assumed for D and L. The assumption that D and L are constant would be approximately realized for thermoelectric but not for galvanic forces. Assuming the constancy of D and L we find the following value for P:

$$P = \chi(x) + Ce^{-\frac{4\pi L}{D}t}, \quad\text{...................(15)}$$

where $\chi(x)$ is to be put equal to zero in those parts of the cylinder, where no external forces reside.

We have seen on p. 142 that $e^{-\frac{4\pi L}{D}}$ is the rate at which the free electricity vanishes from the interior of a conductor, being very rapid for good conductors. In this expression (15) for P the term containing the factor $e^{-\frac{4\pi L}{D}}$ will therefore vanish with exactly this velocity, and may thus be rejected after the elapse of a sufficiently

long period; conversely, its vanishing characterizes the stationary state of the ether. Equation (15) thus becomes

$$P=\chi(x) \text{ for } 0<x<a, \quad P=0 \text{ for } 0>x>a.$$

Since $P=-\dfrac{d\phi}{dx}$, we have for $x<0$

$$-\frac{d\phi}{dx}=0,$$

hence $\phi=\phi_0$, where ϕ_0 is a constant; for $0<x<a$

$$-\frac{d\phi}{dx}=\chi(x),$$

hence $$\phi=-\int\chi(x)dx+\text{const.}=\mathrm{X}(x)+\text{const.},$$

where $$\mathrm{X}(x)=-\int\chi(x)dx, \quad\quad (16)$$

which for $x=0$ gives $\phi_0=\text{const.}$,

hence $$\phi=\mathrm{X}(x)+\phi_0;$$

for $x=a$ this value of ϕ becomes

$$\phi=\mathrm{X}(a)+\phi_0=\phi_1,$$

where ϕ_1 is a constant; for $x>a$

$$\phi=\phi_1.$$

We see, therefore, that as we pass through the region within which the electromotive forces reside, the electrostatic potential ϕ increases by a constant $\mathrm{X}(a)$ determined by formula (16), and that, in the meantime, this region hereby becomes charged with real electricity, whose density is

$$\epsilon_r=-\frac{D}{4\pi}\frac{d}{dx}(\chi(x));$$

in order that ϵ_r may remain finite $\chi(x)$ must therefore be a continuous function. It follows from this value

for ϵ_r that the total quantity of real electricity in the region $x=0$ to $x=a$ is

$$-\frac{DQ}{4\pi}\int\frac{d\chi(x)}{dx}dx=-\frac{DQ}{4\pi}[\chi(a)-\chi(0)]=0,$$

since $\chi(a)=\chi(0)=0$.

If a is very small, that is, if the region $x=0$ to $x=a$ is a mere film or so-called double film, the electromotive force will be given by the following integral:

$$\int_0^a\chi(x)\,dx,$$

and the condition that the conductor be very long—this condition was introduced above in order that all quantities might be regarded as functions of x and t only—will no longer be necessary, but it will then suffice if its dimensions are only large compared to those of a. These equations will also approximately hold when the double film has the form of a curved surface, the normal n to the surface taking the place of the x-axis.

Under the special assumption that χ is constant, $X=-\chi_0$, throughout the region $x=0$ to $x=a$, $\mathrm{X}(a)$ will have the value $-a\chi_0$ and the electrostatic potential ϕ at any point of it will thus be

$$\phi=\phi_0+x\chi_0.$$

In this special case the electrostatic potential ϕ will therefore decrease by the constant $a\chi_0$ in passing through the region of the external electromotive forces, that is, the decrement of ϕ in passing through this region will be directly proportional not only to the electromotive force X, which is constant throughout the region, but to its breadth a. Here ϵ_r would vanish at every point within the region $x=0$ to $x=a$, since

$$\frac{d\chi(x)}{dx}=a\frac{d\chi_0}{dx}=0,$$

whereas on its dividing-surfaces, which we always regard as transition-films,

$$E_r = \pm\frac{D}{4\pi}\int_0^\delta \frac{d}{dx}(\chi(x))dx = \pm\frac{aD}{4\pi}\chi_0.$$

In the above problem we have not only assumed given values for X, Y, Z—$X=-\chi(x)$, $Y=Z=0$—but supposed that these given forces are confined to a given region of a conductor of given configuration. Let us next examine the most general case, that is, the most general solution of equations (12).

Assuming D and L to be constant, differentiating equations (12), the first with regard to x, the second to y, and the third to z, and adding we find the following conditional equation between ϕ and ψ,

$$\frac{D}{4\pi L}\frac{d}{dt}\nabla^2\phi + \nabla^2\phi = \nabla^2\psi, \quad \text{..............}(17)$$

where ψ is a given function of x, y, z. This equation is of the same form as equation (13)—here $\nabla^2\phi$ is to be regarded as the variable, since $\nabla^2\psi$ does not contain the time; it can therefore be integrated by Euler's formula (14), and we get

$$\nabla^2\phi = \nabla^2\psi + Ce^{-\frac{4\pi L}{D}t}, \quad \text{.................}(18)$$

where the arbitrary constant C would have to be determined as a function of x, y, z from the initial conditions. For reasons similar to those on p. 142, etc., the term $Ce^{-\frac{4\pi L}{D}t}$ will finally vanish, and on its disappearance the electrostatic state or, as we shall designate it in this general case, the stationary state, will set in (see also next article). As long as this term remains perceptible, the state of the ether though aphotic cannot become stationary. For very large values of L the beginning of the stationary state will coincide approximately with

that of the aphotic. The former is here characterized by the equation

$$\nabla^2\phi=\nabla^2\psi, \qquad\qquad (19)$$

where D and L have been assumed constant.

If D and L are variable, we have the following equation for the aphotic state of the ether instead of equation (17):

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d\left(D\frac{d\phi}{dx}\right)}{dx}+\frac{d\left(D\frac{d\phi}{dy}\right)}{dy}+\frac{d\left(D\frac{d\phi}{dz}\right)}{dz}\right]$$
$$+\frac{d}{dx}\left(L\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi}{dz}\right)$$
$$=\frac{d}{dx}\left(L\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi}{dz}\right), \qquad (20)$$

and instead of equation (19) the following for its stationary state:

$$\frac{d}{dx}\left(L\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi}{dz}\right)$$
$$=\frac{d}{dx}\left(L\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi}{dz}\right) \qquad (21)$$

At any point on the surface of the conductor the stationary state will be characterized by the equation

$$\frac{d\phi}{dn}=\frac{d\psi}{dn}. \qquad\qquad (22)$$

For the derivation of these last two more general equations we are obliged to refer the student to the next article.

Let us next investigate the stationary state of the given conductor. The external electromotive forces

$$X=\frac{d\psi}{dx},\quad Y=\frac{d\psi}{dy},\quad Z=\frac{d\psi}{dz},$$

will in general be confined to a given region of the conductor; let us call this region for brevity the *electromotive region.* All the other regions of the conductor, those, namely, where $X = Y = Z = 0$, we shall include under the name of the *non-electromotive region.* In the latter we have

$$\frac{d}{dx}\left(L\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi}{dz}\right)=0,$$

or, if the conductor is homogeneous,

$$\nabla^2\psi=0,$$

hence by the above condition (21)

$$\left.\begin{array}{l}\dfrac{d}{dx}\left(L\dfrac{d\phi}{dx}\right)+\dfrac{d}{dy}\left(L\dfrac{d\phi}{dy}\right)+\dfrac{d}{dz}\left(L\dfrac{d\phi}{dz}\right)=0,\\ \text{or, if } L \text{ is constant,} \qquad \nabla^2\phi=0.\end{array}\right\}\ \ldots\ldots(23)$$

Designating the above system as the system A, let us compare it with a second system B, which differs from the former in only the one respect that the external electromotive forces of system A are wanting in system B. In the ensuing comparison we must now discriminate between the three following systems:

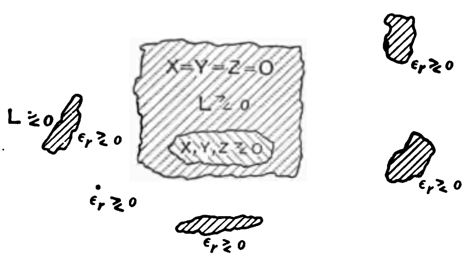

FIG. 12.

(1) The electromotive region is contained entirely within the conductor, as indicated in figure 12.

(2) The electromotive region extends to the surface of the conductor, including a piece of that surface, figure 13.

(3) The conductor is divided into two separate regions by the electromotive region, figure 14.

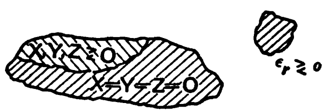

FIG. 13.

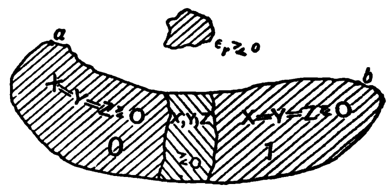

FIG. 14.

Comparing systems $1A$ with $1B$ we see that

$$\phi_B = \text{const.} = \Phi$$

(cf. p. 119); for the non-electromotive region

$$\psi = \text{const.} = \Psi.$$

$$\phi_A = \Phi + \Psi$$

will then be the solution for ϕ in the non-electromotive region of system A. This same relation will also hold for the electromotive region, if we regard Ψ as a given function of x, y, z instead of as a constant. The general solution for ϕ_A can therefore be written in the form

$$\phi_A = \Phi + \psi, \qquad\text{(24)}$$

where ψ is to be assigned the constant value Ψ at any point, where X, Y, Z vanish, whereas at any point of the electromotive region the value of ψ at that point is to be taken.

That the above expression (24) is a solution for ϕ_A is evident, since both conditions (21) and (22) are fulfilled; that it is the only solution follows from the following theorem from the theory of the potential: "If at every point in the interior of a conductor

$$\frac{d}{dx}\left(L\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi}{dz}\right)$$
$$=\frac{d}{dx}\left(L\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi}{dz}\right),$$

at every point on its surface

$$\frac{d\phi}{dn}=\frac{d\psi}{dn},$$

moreover, if the derivatives of ϕ are finite and continuous throughout the given region (conductor), and ϕ vanishes at infinity, ϕ is uniquely determined." This is only a special form of the general theorem from the theory of the potential proved in the second article of the next chapter. The continuity and finiteness of the derivatives of ϕ are taken for granted here for reasons similar to those already mentioned (cf. pp. 104-105).

The comparison of systems $2A$ with $2B$ and $3A$ with $3B$ is exactly similar to the above, and leads in the former case to the same solution (24) and in the latter to the following expressions for ϕ_A:

$$\left.\begin{aligned}\phi_{A0}&=\Phi+\Psi_0\\ \phi_{A1}&=\Phi+\Psi_1\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

where the index 0 or 1 suffixed to ϕ and ψ indicates the value of the given quantity in the one or the other non-electromotive region respectively of the conductor, that is, in that, to which 0 or 1 respectively is affixed in figure 14; in the electromotive region the value of ψ at the given point is to be substituted for the constant values Ψ_0 and Ψ_1.

By the above solution for ϕ_A the density of the real electricity at any point of the conductor can be written

$$\epsilon_r=-\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\phi_A}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi_A}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi_A}{dz}\right)\right]$$
$$=-\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\psi}{dz}\right)\right].$$

Let us next compare the expressions for the free electricity on the surface of the conductor of systems A and B. According to equation (23, VI.) the surface-density of the free electricity at any point on the surface of the conductor of system $1A$ is

$$E_f = \frac{1}{4\pi}\left(\frac{d\phi_A}{dn} - \frac{d\phi}{dn}\right), \quad (\epsilon = 1)$$

where ϕ (without the index) denotes the value of the electrostatic potential in the surrounding medium, or since $\frac{d\phi_A}{dn}$ vanishes by formula (24),

$$E_f = -\frac{1}{4\pi}\frac{d\phi}{dn};$$

at any point on the surface of the conductor of system $1B$ we find an analogous expression. We should observe that although ϕ refers to the value of the electrostatic potential in the surrounding medium of either system A or B, its value in the one system will not necessarily be the same as in the other. In the present case, however, ϕ happens to have the same value in both systems. To prove this, let us compare system A with a third system C, which differs from the former in only two respects, namely, that its electromotive region is connected with the earth and that there are no other bodies in the surrounding medium (dielectric) but the given conductor. ϕ_C will then evidently be constant at every point of the dielectric; at any point of the conductor the following condition (21) must hold:

$$\frac{d}{dx}\left(L\frac{d\phi_C}{dx}\right) + \frac{d}{dy}\left(L\frac{d\phi_C}{dy}\right) + \frac{d}{dz}\left(L\frac{d\phi_C}{dz}\right)$$

$$= \frac{d}{dx}\left(L\frac{d\psi}{dx}\right) + \frac{d}{dy}\left(L\frac{d\psi}{dy}\right) + \frac{d}{dz}\left(L\frac{d\psi}{dz}\right),$$

and at any point on its surface the following (22):

$$\frac{d\phi_C}{dn}=\frac{d\psi}{dn},$$

which reduces to
$$\frac{d\phi_C}{dn}=0,$$

since ψ vanishes within the given (non-electromotive) region.

We know now from the theorem just stated that ϕ_C will be uniquely determined within the conductor by these last two conditions; it follows, therefore, from the first condition, $\phi_C=constant$ at every point of the dielectric, that ϕ_C will be uniquely determined at every point of space.

The following conditions will evidently hold for system A at any point of the conductor:

$$\frac{d}{dx}\left(L\frac{d\phi_A}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi_A}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi_A}{dz}\right)$$
$$=\frac{d}{dx}\left(L\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi}{dz}\right),$$

at any point on its surface

$$\frac{d\phi_A}{dn}=\frac{d\psi}{dn}=0,$$

and at any point of the surrounding medium

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d}{dx}\left(D\frac{d\phi_A}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi_A}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi_A}{dz}\right)\right]$$
$$+\frac{d}{dx}\left(L\frac{d\phi_A}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi_A}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi_A}{dz}\right)=0.$$

Since ϕ_A is uniquely determined by these three conditions,* it follows that any function that satisfies them

---

* Cf. § 21.

will be the desired solution. Such a function is now

$$\phi_A = \phi_B + \phi_C, \qquad\qquad (26)$$

where ϕ_B is the solution of the above system B, which is determined uniquely by the conditions

$$\frac{d}{dx}\left(L\frac{d\phi_B}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi_B}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi_B}{dz}\right)=0$$

at any point within the conductor,

$$\frac{d\phi_B}{dn}=0$$

at any point on its surface and

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d}{dx}\left(D\frac{d\phi_B}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi_B}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi_B}{dz}\right)\right]$$
$$+\frac{d}{dx}\left(L\frac{d\phi_B}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi_B}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi_B}{dz}\right)=0$$

at any point of the surrounding medium; and from this solution (26) for ϕ_A it follows that

$$E_{fA}=-\frac{1}{4\pi}\frac{d\phi_A}{dn}=-\frac{1}{4\pi}\frac{d\phi_B}{dn},$$

since $\dfrac{d\phi_C}{dn}=0$ at any point of the dielectric. Q.E.D.

The electromotive forces in the given case, where namely they are confined to the interior of the conductor, will manifest themselves on its surface only at the moment of their creation, when the quantity of real electricity

$$\frac{1}{4\pi}\int\left\{\frac{d}{dx}\left(D\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\psi}{dz}\right)\right\}d\tau,$$

where the integration is to be extended through the electromotive region, is drawn into the electromotive

region to neutralize the action of the external forces themselves. After the ether has subsided to the stationary state, the non-electromotive region will therefore act, so to speak, as a screen on these forces. It is evident, however, that this will only be the case when the external electromotive forces have a potential.

The expressions for the free electricity in case 2 at any point of the electromotive region on the surface of the conductor will be

$$E_{fA}=\frac{1}{4\pi}\left(\frac{d\psi}{dn}-\frac{d\phi}{dn}\right) \quad \text{and} \quad E_{fB}=-\frac{1}{4\pi}\frac{d\phi}{dn},$$

and at any other point on its surface

$$E_f=-\frac{1}{4\pi}\frac{d\phi}{dn}$$

for both systems, A and B, where, as above, ϕ denotes the electrostatic potential in the surrounding medium. Here the density of the free electricity at every point of the electromotive region on the surface of the conductor evidently increases by the quantity $\frac{1}{4\pi}\frac{d\phi}{dn}$ as we pass from systems B to A. It thus follows that the value of ϕ at any point of the surrounding medium will no longer remain the same in both systems. As a rule the electromotive region is, however, either so small in comparison to the regions under consideration, or at such a distance from them that the effect of the slight variation of E_f in the former on the value of ϕ in the latter may be entirely overlooked. In case 3 we can also generally neglect this same variation in ϕ, as we pass from the one system to the other; in fact, with the exception of the constant difference of potential produced by the presence of the electromotive region, all other effects due to its presence can be overlooked in most problems of electrostatics. A galvanic element, whose terminals are connected with two large metallic

spheres or plates at a considerable distance apart, may be taken as an example of such a system.

The expressions for E_f in case 3, systems A and B, will of course be similar to those already found for E_f in case 2.

A special form of case 3 is that where the two separate non-electromotive regions 0 and 1 of our conductor, instead of being entirely insulated as in figure 14—we exclude here the connection by the electromotive region —are connected with each other, as indicated in the annexed figure. The stationary state of the ether in such an annular conductor will of course be characterized

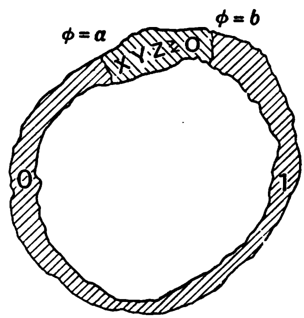

FIG. 15.

by the same equations (21) and (22) as above, but the ensuing equations and integrals will no longer hold; this follows, since ϕ can evidently no longer remain constant within the non-electromotive region of the conductor, but must change its value from $\phi = a$ on the one dividing-surface between the electromotive and non-electromotive regions to $\phi = b$ on its other—its rate of change will of course depend on the configuration and constitution of the non-electromotive region of the conductor (cf. § 22); here, P, Q, R cannot therefore vanish, but must either remain constant as in linear homogeneous conductors or become functions of x, y, z. The state of

the ether within the conductor cannot thus be designated as electrostatic or stationary, as above; on the other hand, the ether is said in this case to have subsided to the state of stationary flow. The solution of equations (21) and (22) for the more general case of stationary flow, where namely X, Y, Z are not required to have a potential, is given in the next chapter. It is, however evident that the solution of the given problem can be written in the form

$$\phi = \psi - k\psi', \qquad\qquad (27)$$

where ψ' is determined by the conditions that at every point within the conductor

$$\frac{d}{dx}\left(L\frac{d\psi'}{dx}\right) + \frac{d}{dy}\left(L\frac{d\psi'}{dy}\right) + \frac{d}{dz}\left(L\frac{d\psi'}{dz}\right) = 0,$$

at every point on its surface

$$\frac{d\psi'}{dn} = 0,$$

and at every point on either dividing-surface between the electromotive and non-electromotive regions it has a given constant value—the difference between these two values of ψ' has been taken here as unity; ψ is thus a function of the configuration only of the conductor; k is a constant. If L is constant, ψ' may be conceived as the velocity potential of an incompressible fluid flowing in a closed channel without rotation. As this solution (27) fulfils all conditions for ϕ, not only the conditional equations (21) and (22) but the condition that ϕ changes its value by a constant k in passing from the one dividing-surface between the electromotive and non-electromotive regions to the other, it is evidently its only solution (cf. theorem proved in § 21). Here, as in the preceding cases, there will be an accumulation of real electricity in the interior and on the surface of the conductor, which, although it will not completely

counteract the action of the external electromotive forces, for

$$\epsilon_r = -\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\psi}{dz}\right)\right]$$
$$+\frac{k}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\psi'}{dx}\right)+\frac{d}{dy}\left(D\frac{d\psi'}{dy}\right)+\frac{d}{dz}\left(D\frac{d\psi'}{dz}\right)\right],$$

will nevertheless be such that the current-strength at every point of the conductor will depend only on the value of the constant k, the potential difference between the two terminals, and the configuration of the conductor itself; this follows, moreover, from formulae (13, III.), and the above value (27) for ϕ, since

$$p = L\left(P+\frac{d\psi}{dx}\right) = L\left(-\frac{d\phi}{dx}+\frac{d\psi}{dx}\right) = kL\frac{d\psi'}{dx},$$
$$q = kL\frac{d\psi'}{dy}, \quad r = kL\frac{d\psi'}{dz}.$$

We shall not examine this special case further here, but refer the student to the treatment of the more general case in the next chapter.

CHAPTER IX.

SECTION XX. SUBSIDENCE OF THE ETHER TO THE STATE OF STATIONARY FLOW, X, Y, Z, INDEPENDENT OF THE TIME.

In the last article we assumed that the external electromotive forces X, Y, Z had a potential, and examined the so-called stationary state of the ether, which in homogeneous conductors was characterized by the vanishing of the term $Ce^{-\frac{4\pi L}{D}t}$. Let us next prove that the ether will finally subside to the stationary state or that of stationary flow, as the case may be, in any system of conductors, homogeneous or non-homogeneous, and whether the external electromotive forces have a potential or not, provided they are only independent of the time.

In addition to the differential equations (9, II.) and (10, II.) for

$$P=P(t), \quad Q=Q(t), \quad \dots \quad a=a(t), \dots$$

let the initial conditions ($t=0$) also be given; these may be designated symbolically as the initial conditions A and written

$$P=P(o), \quad Q=Q(o), \quad \dots \quad a=a(o). \qquad \text{(1)}$$

All these quantities will, of course, be functions of the coordinates, which have been omitted for brevity. If we denote by t_1 any positive quantity, the conditions that correspond to $t=t_1$ will be

$$P=P(t_1), \quad Q=Q(t_1), \quad \dots \quad a=a(t_1) \dots .$$

Instead of equations (1) we could now regard these equations as our initial conditions, designating them symbolically as the initial conditions B; after an elapse of the time t we should then have

$$P=P(t+t_1),\quad Q=Q(t+t_1),\ \dots\ a=a(t+t_1),\dots.$$

Since we have assumed that the external electromotive forces X, Y, Z do not change with the time, the X, Y, Z of the one system, $t=t_1$ will be the same as those of the other $t=t+t_1$; in subtracting the quantities of the one system from those of the other we thus obtain a third system,

$$P=P(t+t_1)-P(t),$$

$$Q=Q(t+t_1)-Q(t),\ \dots\ a=a(t+t_1)-a(t)\ \dots,$$

in which X, Y, Z will vanish.

As our fundamental differential equations are linear, these values for P, Q, R, a, β, γ will also satisfy our fundamental equations. They are in fact the solutions of equations (9, II.) and (10, II.), when $X=Y=Z=0$, and the initial values of P, Q, R, a, β, γ are the differences between the values of the respective quantities corresponding to conditions B and A. We have already seen on p. 119 that in any such system of conductor P, Q, R will finally vanish. Consequently for large values of t the following equations must always hold:

$$P=P(t+t_1)-P(t)=0,\quad Q=Q(t+t_1)-Q(t)=0\ \dots.$$

These will be valid for all conductors, provided we only choose a sufficiently large value for t. Moreover, since these equations must hold for every value of t_1, provided the value of t is only chosen large enough, it follows that P, Q, R (in conductors) will become constant after the elapse of a sufficiently long period. We shall call this state of the ether that of stationary flow, and regard the stationary and electrostatic states as special cases of it.

The only difference between electrostatic phenomena and those of stationary flow—we shall include the stationary state, as special case, under this category in the following—is that in conductors for the former $P=Q=R=0$, whereas for the latter they require only to be independent of the time. Hence by equations (14, V.) the derivatives of ϕ with regard to the coordinates must also be independent of the time. It follows from equation (2, III.) that ϵ_r will be independent of the time in insulators not only for electrostatic phenomena but for those of stationary flow and from equation (1, III.) that ϵ_r will also be independent of the time in conductors, since D, P, Q, R are here functions of the coordinates only; the same will also, of course, hold for E_r, since the surface-elements of adjoining media can always be treated as volume-elements.

If the use of equation (1, III.) is objected to on account of any doubts about the constancy of D in conductors, we can have recourse to equation (2, III.) for the interior of conductors, and to equation (9, III.) for the dividing-surfaces between conductors and insulators, from which it follows that $\frac{dr_\epsilon}{dt}$ and $\frac{dE_r}{dt}$ will be independent of the time. These quantities must now finally ($t=\infty$) vanish, since the tonic motion would otherwise never cease, that is, since the head of energy would otherwise have to be assumed infinitely large. Consequently we must have

$$\frac{d\epsilon_r}{dt}=\frac{dE_r}{dt}=0,$$

hence $\qquad \epsilon_r=f(x,y,z), \quad E_r=F(x,y,z).$

Equations (2, III.) and (9, III.) can therefore be written

$$\left.\begin{array}{c}\frac{d}{dx}L(P+X)+\frac{d}{dy}L(Q+Y)+\frac{d}{dz}L(R+Z)=0\\ L_1(N_1+S_1)=L_0(N_0-S_0),\end{array}\right\},\ldots(2)$$

where N and S are the components of the vectors (P, Q, R) and (X, Y, Z) respectively along the normal to the surface at the given point.

We could have obtained the former of these equations directly by differentiating equations (16, V.) the first with regard to x, the second to y, and the third to z, and by adding.

Introducing ϕ into equations (2), we have

$$\frac{d}{dx}L\left(\frac{d\phi}{dx}-X\right)+\frac{d}{dy}L\left(\frac{d\phi}{dy}-Y\right)+\frac{d}{dz}L\left(\frac{d\phi}{dz}-Z\right)=0, \ldots\text{(3)}$$

and
$$L_1\left(\frac{d\phi_1}{dn}-S_1\right)=L_0\left(\frac{d\phi_0}{dn}-S_0\right). \ldots\ldots\text{(4)}$$

If the vector S is continuous in the transition-film, $S_1=S_0$ approximately (cf. p. 118), and relation (4) reduces to

$$L_1\frac{d\phi_1}{dn}=L_0\frac{d\phi_0}{dn}. \ldots\ldots\text{(5)}$$

If the dividing-surface is that between an insulator and a conductor, $L=S=0$ for the former, and hence the following relation will hold for the latter:

$$\frac{d\phi}{dn}-S=0, \ldots\ldots\text{(6)}$$

where for simplicity we have dropped the index; or in case S may be rejected

$$\frac{d\phi}{dn}=0. \ldots\ldots\text{(7)}$$

In the interior of non-conductors and on their dividing-surfaces ϕ will be determined by the equations we have already found for electrostatics, namely,

$$\epsilon_r=-\frac{1}{4\pi}\left[\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)\right] \ldots\text{(8)}$$

and
$$E_r=\frac{1}{4\pi}\left(D_0\frac{d\phi_0}{dn}-D_1\frac{d\phi_1}{dn}\right), \ldots\ldots\text{(9)}$$

where ϵ_r and E_r are the respective densities of the real electricity at the given point. On the dividing-surface between conductors and insulators formula (9) becomes

$$E_r = -\frac{1}{4\pi} D_1 \frac{d\phi_1}{dn},$$

where the normal n is to be drawn from the conductor into the insulator. The total quantity of electricity on the surface of any insulated conductor will therefore be

$$\int E_r do = -\frac{1}{4\pi}\int D_1 \frac{d\phi_1}{dn} do.$$

Lastly, we should observe that ϕ will not in general be discontinuous on the dividing-surfaces of adjoining bodies, that is, within their transition-films, since its derivatives with regard to the coordinates, that is, the tonic motions themselves, would then become infinite. This could only be the case, when the external electromotive forces were infinitely large in the dividing-surfaces; this is taken for granted in the theory of electricity, whenever a so-called difference of potential is assumed on the surface of contact of two bodies. As, however, we always consider such a case as the limit of that where very large electromotive forces are distributed through a very thin film—we shall return to this subject in § 22 (cf. p. 178)—we can thus assume the continuity of ϕ throughout space.

SECTION XXI. PROOF OF THE UNIQUE DETERMINATION OF ϕ*.

ϕ is uniquely determined by the following conditions:

$$(1)\ \frac{d}{dx}L\left(\frac{d\phi}{dx}-X\right)+\frac{d}{dy}L\left(\frac{d\phi}{dy}-Y\right)+\frac{d}{dz}L\left(\frac{d\phi}{dz}-Z\right)=0 \quad ..(10)$$

* Cf. Riemann's *Vorlesungen über Schwere, Electricität und Magnetismus*, §§ 57 and 58.

within all conductors,

(2) $$L_1\left(\frac{d\phi_1}{dn}-S_1\right)=L_0\left(\frac{d\phi_0}{dn}-S_0\right) \qquad\qquad (11)$$

on the dividing-surface of two conductors,

(3) $$\frac{d\phi}{dn}-S=0 \qquad\qquad (12)$$

on the dividing-surface between any conductor and an insulator, and (4) the continuity and finiteness of the first derivatives of ϕ in the regions under consideration.

We exclude the transition-films between the conductors of the system from the regions under consideration and examine its remaining regions, which we shall designate by S. At any point of the latter X, Y, Z and L will be finite and continuous. Let V be any function of x, y, z that satisfies the two following conditions: (1) the difference between the values of V at any two points of a transition-film, that lie diametrically opposite, is given and is finite; (2) the derivatives of V at any point in the regions S are finite and continuous. There is now an infinite number of such functions V. If we denote by V any one of these functions, any other such function—let us denote it by v—must then evidently have the form

$$v=V+h\mathfrak{s},$$

where h is an arbitrary constant and $\mathfrak{s}$ is a function of x, y, z, which is finite and continuous not only throughout the regions S but in the transition-films themselves. The derivatives of V in the latter regions may, however, either change very rapidly or be infinite and discontinuous; these are identical to the assumptions already made for ϕ (cf. pp. 105 and 167).

The integral

$$\Omega(v)=\int L\left\{\left(\frac{dv}{dx}-X\right)^2+\left(\frac{dv}{dy}-Y\right)^2+\left(\frac{dv}{dz}-Z\right)^2\right\}dS, \quad (13)$$

the integration being extended through the regions S, will evidently have a finite positive value for all functions v. Among these functions v let there be at least one—we shall denote it by V—for which the value of this integral is a minimum. The condition for such a minimum is now

$$\Omega(V) \leqq \Omega(V+h\mathfrak{s}), \qquad \text{.....................(14)}$$

when h is taken infinitely small. The right-hand side of this inequality can be developed as a function of V and $\mathfrak{s}$. We have

$$\frac{d}{dx}(V+h\mathfrak{s})=\frac{dV}{dx}+h\frac{d\mathfrak{s}}{dx}, \quad \frac{d}{dy}(V+h\mathfrak{s})=\frac{dV}{dy}+h\frac{d\mathfrak{s}}{dy}$$

$$\frac{d}{dz}(V+h\mathfrak{s})=\frac{dV}{dz}+h\frac{d\mathfrak{s}}{dz};$$

hence

$$\left[\frac{d}{dx}(V+h\mathfrak{s})-X\right]^2+\left[\frac{d}{dy}(V+h\mathfrak{s})-Y\right]^2+\left[\frac{d}{dz}(V+h\mathfrak{s})-Z\right]^2$$
$$=\left[\left(\frac{dV}{dx}-X\right)^2+\left(\frac{dV}{dy}-Y\right)^2+\left(\frac{dV}{dz}-Z\right)^2\right]$$
$$+2h\left[\left(\frac{dV}{dx}-X\right)\frac{d\mathfrak{s}}{dx}+\left(\frac{dV}{dy}-Y\right)\frac{d\mathfrak{s}}{dy}+\left(\frac{dV}{dz}-Z\right)\frac{d\mathfrak{s}}{dz}\right]$$
$$+h^2\left[\left(\frac{d\mathfrak{s}}{dx}\right)^2+\left(\frac{d\mathfrak{s}}{dy}\right)^2+\left(\frac{d\mathfrak{s}}{dz}\right)^2\right],$$

from which we find

$$\Omega(V+h\mathfrak{s})=\Omega(V)$$
$$+2h\int L\left[\left(\frac{dV}{dx}-X\right)\frac{d\mathfrak{s}}{dx}+\left(\frac{dV}{dy}-Y\right)\frac{d\mathfrak{s}}{dy}+\left(\frac{dV}{dz}-Z\right)\frac{d\mathfrak{s}}{dz}\right]dS$$
$$+h^2\int L\left[\left(\frac{d\mathfrak{s}}{dx}\right)^2+\left(\frac{d\mathfrak{s}}{dy}\right)^2+\left(\frac{d\mathfrak{s}}{dz}\right)^2\right]dS. \qquad \text{.....................(15)}$$

The first and third terms of this development (15) are evidently positive, whereas its second term can be either

positive or negative. To satisfy the above condition (14), the only and necessary condition will therefore be

$$\int L\left[\left(\frac{dV}{dx}-X\right)\frac{d\mathfrak{s}}{dx}+\left(\frac{dV}{dy}-Y\right)\frac{d\mathfrak{s}}{dy}+\left(\frac{dV}{dz}-Z\right)\frac{d\mathfrak{s}}{dz}\right]dS=0. \quad (16)$$

That this conditional equation is not only sufficient, but, on the other hand, necessary, is evident, since, if it did not hold, the sign of h could be so chosen, that the second term of equation (15) would become negative, and the absolute value of h could then be taken so small that the value of the third term would be smaller than that of the second, and we should have

$$\Omega(V+h\mathfrak{s})<\Omega(V).$$

Equation (16) can be transformed in the following manner:

$$\int\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)dS=\int\left(f\frac{dx}{dn}+g\frac{dy}{dn}+h\frac{dz}{dn}\right)d\sigma, \quad \ldots(17)$$

where dS is a volume-element, $d\sigma$ a surface-element, and

$$\left.\begin{aligned} f&=L\left(\frac{dV}{dx}-X\right)\mathfrak{s}\\ g&=L\left(\frac{dV}{dy}-Y\right)\mathfrak{s}\\ h&=L\left(\frac{dV}{dz}-Z\right)\mathfrak{s}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

Thus

$$\begin{aligned}&\int L\left[\left(\frac{dV}{dx}-X\right)\frac{d\mathfrak{s}}{dx}+\left(\frac{dV}{dy}-Y\right)\frac{d\mathfrak{s}}{dy}+\left(\frac{dV}{dz}-Z\right)\frac{d\mathfrak{s}}{dz}\right]dS\\ &\quad+\int\mathfrak{s}\left[\frac{d}{dx}L\left(\frac{dV}{dx}-X\right)+\frac{d}{dy}L\left(\frac{dV}{dy}-Y\right)+\frac{d}{dz}L\left(\frac{dV}{dz}-Z\right)\right]dS\\ &\quad=\int\mathfrak{s}L\left[\left(\frac{dV}{dx}-X\right)\frac{dx}{dn}+\left(\frac{dV}{dy}-Y\right)\frac{dy}{dn}+\left(\frac{dV}{dz}-Z\right)\frac{dz}{dn}\right]d\sigma;\end{aligned}$$

and equation (16) can be written

$$\left.\begin{aligned}&\int\varepsilon\left[\frac{d}{dx}L\left(\frac{dV}{dx}-X\right)+\frac{d}{dy}L\left(\frac{dV}{dy}-Y\right)+\frac{d}{dz}L\left(\frac{dV}{dz}-Z\right)\right]dS\\&=\int\varepsilon L\left[\left(\frac{dV}{dx}-X\right)\frac{dx}{dn}+\left(\frac{dV}{dy}-Y\right)\frac{dy}{dn}+\left(\frac{dV}{dz}-Z\right)\frac{dz}{dn}\right]d\sigma,\end{aligned}\right\}(19)$$

where the first integration is to be extended through the regions S, and the second over their surfaces. It is evident that equation (19) can only be satisfied by putting each integral equal to zero, since the one integral is entirely independent of the other. The volume-integral will vanish, if the expression

$$\frac{d}{dx}L\left(\frac{dV}{dx}-X\right)+\frac{d}{dy}L\left(\frac{dV}{dy}-Y\right)+\frac{d}{dz}L\left(\frac{dV}{dz}-Z\right)=0$$

at every point in the interior of regions S. This equation is now our first condition (10) for ϕ.

The surfaces of the regions S consist of the surfaces proper of the conductors and those of the transition-films between adjoining conductors. If we put

$$\left(\frac{dV}{dx}-X\right)\frac{dx}{dn}+\left(\frac{dV}{dy}-Y\right)\frac{dy}{dn}+\left(\frac{dV}{dz}-Z\right)\frac{dz}{dn}=0$$

at every point on the former surfaces, the surface-integral of condition (19) extended over these surfaces will vanish. This equation is now our third condition (12) for ϕ.

The surface-integral of condition (19), extended over any given transition-film, can evidently be written in the form

$$\int\left\{\begin{aligned}&\varepsilon_0L_0\left[\left(\frac{dV_0}{dx}-X_0\right)\frac{dx}{dn}+\left(\frac{dV_0}{dy}-Y_0\right)\frac{dy}{dn}+\left(\frac{dV_0}{dz}-Z_0\right)\frac{dz}{dn}\right]\\-&\varepsilon_1L_1\left[\left(\frac{dV_1}{dx}-X_1\right)\frac{dx}{dn}+\left(\frac{dV_1}{dy}-Y_1\right)\frac{dy}{dn}+\left(\frac{dV_1}{dz}-Z_1\right)\frac{dz}{dn}\right]\end{aligned}\right\}d\sigma,(20)$$

where the two similar expressions under the integral sign, the one with the index 1 and the other with the

index 0, denote the values of the given expression at any two points on the surface of the film that lie on the same normal. As in the above form of Green's law (17), the normal to the transition-film (surface of discontinuity) is always to be drawn inwards, that is, here into the regions S, and as every quantity is to be referred to this inward normal, these two expressions will have opposite signs, when the terms of both expressions are referred to the same normal; this is the reason for the negative sign before the second expression of our integral (20).

As we pass through the transition-film, the function $\mathfrak{s}$ varies continuously, that is, it varies very slowly in comparison to the derivatives of ϕ, or to the external electromotive forces X, Y, Z, as we have seen above; $\mathfrak{s}_1 - \mathfrak{s}_0$ will, therefore, be very small in comparison to the difference between the value of any other quantity of this integral (20) taken at any point on the surface of the film, and that of the same quantity at the corresponding point on its opposite surface; for instance, $\mathfrak{s}_1 - \mathfrak{s}_0$ will be very small in comparison to

$$\frac{dV_1}{dx} - \frac{dV_0}{dx} \quad \text{or} \quad X_1 - X_0,$$

provided X is discontinuous or varies very rapidly in the transition-film; on the other hand, if X, Y, Z are continuous or vary slowly in the film, they can then be rejected in comparison to $\frac{dV}{dx}$, $\frac{dV}{dy}$, and $\frac{dV}{dz}$ respectively.

We can thus put $\mathfrak{s}_1 = \mathfrak{s}_0$ in integral (20), and we have

$$\int\!\!\int \left\{ \begin{aligned} & L_0\left[\left(\frac{dV_0}{dx} - X_0\right)\frac{dx}{dn} + \left(\frac{dV_0}{dy} - Y_0\right)\frac{dy}{dn} + \left(\frac{dV_0}{dz} - Z_0\right)\frac{dz}{dn}\right] \\ -& L_1\left[\left(\frac{dV_1}{dx} - X_1\right)\frac{dx}{dn} + \left(\frac{dV_1}{dy} - Y_1\right)\frac{dy}{dn} + \left(\frac{dV_1}{dz} - Z_1\right)\frac{dz}{dn}\right] \end{aligned} \right\} \mathfrak{s}\, d\sigma. \quad (21)$$

By equation (19) this integral extended over all the transition-films of the regions S must vanish. This condition can now be fulfilled by putting

$$L_0\left[\left(\frac{dV_0}{dx}-X_0\right)\frac{dx}{dn}+\left(\frac{dV_0}{dy}-Y_0\right)\frac{dy}{dn}+\left(\frac{dV_0}{dz}-Z_0\right)\frac{dz}{dn}\right]$$
$$=L_1\left[\left(\frac{dV_1}{dx}-X_1\right)\frac{dx}{dn}+\left(\frac{dV_1}{dy}-Y_1\right)\frac{dy}{dn}+\left(\frac{dV_1}{dz}-Z_1\right)\frac{dz}{dn}\right]$$

for every normal to every dividing-surface of the adjoining conductors. This equation is our second condition (11) for ϕ.

We see, therefore, that any function V, for which the integral (13) is a minimum, will satisfy the conditions (1), (2), (3), (4), on pp. 167, 168, and conversely, that for any function that satisfies these conditions this integral will be a minimum. It is now easy to prove that there is only one function V, for which the above integral (13) is a minimum, and hence that there is only one function V, that will satisfy these conditions. For this purpose suppose that any other of the above functions, $v=V+\mathfrak{s}$, is also a minimum of our integral (13); the condition for such a minimum would be

$$\Omega(V+\mathfrak{s})<\Omega(V+h\mathfrak{s}), \quad\text{..................(22)}$$

when the constant h differs from unity by an infinitely small quantity.

The development of $\Omega(V+h\mathfrak{s})$ as a function of V can be found by formula (15) without further difficulty; its second term will vanish in conformity to our conditional equation (16) for V. The development of $\Omega(V+\mathfrak{s})$ follows directly from that for $\Omega(V+h\mathfrak{s})$ by putting $h=1$, its second term, of course, vanishing also.

The above condition (22) for a minimum will, therefore, reduce to

$$\int L\left\{\left(\frac{d\mathfrak{s}}{dx}\right)^2+\left(\frac{d\mathfrak{s}}{dy}\right)^2+\left(\frac{d\mathfrak{s}}{dz}\right)^2\right\}dS$$
$$\leqq h^2\int L\left\{\left(\frac{d\mathfrak{s}}{dx}\right)^2+\left(\frac{d\mathfrak{s}}{dy}\right)^2+\left(\frac{d\mathfrak{s}}{dz}\right)^2\right\}dS \quad\text{..........(23)}$$

As we can choose h not only larger but smaller than unity, this equation can only be fulfilled by the

vanishing of the integral itself, and this can evidently only be effected by putting

$$\frac{ds}{dx}=\frac{ds}{dy}=\frac{ds}{dz}=0, \quad \text{hence } s=\text{const.}$$

at every point in the interior of the regions S.

We see, therefore, that there is only one function V, with the exception of course of the functions $v=V+\text{const.}$, that will satisfy the conditions (1), (2), (3), (4) on pp. 167, 168. The existence of such a function is, of course, taken for granted from the nature of the problem.

From the following familiar theorem from the theory of the potential, we know then that $\phi(V)$ is uniquely determined at every point of space: "If any single-valued function ϕ is given throughout any region S, there is only one function ϕ, that will satisfy the conditions

$$\nabla^2\phi=-4\pi\sigma$$

in the remaining regions of space." The proof of the more general theorem corresponding to the present more general case, where, namely, D is variable, and hence

$$\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)=-4\pi\epsilon_r,$$

—cf. formula (8)—would of course require extensive investigations similar to those that constitute the theory of the potential.

SECTION XXII. EXAMPLES OF THE ABOVE GENERAL CASE OF STATIONARY FLOW. MOST GENERAL FORM OF OHM'S LAW; KIRCHHOFF'S LAWS.

Let us first examine the special case, where any system $\mathfrak{S}$ of connected conductors, within which external electromotive forces act, that are entirely arbitrary except that they are not functions of the time, is brought

in contact (at given points) with a second system $\mathfrak{S}_A$ of connected conductors, within which no external electromotive forces reside. After a sufficiently long period from the moment of their contact, equation (3) will hold at every point in the interior of system $\mathfrak{S}_A$, namely,

$$\frac{d}{dx}\left(L\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi}{dz}\right)=0. \quad \text{.........(24)}$$

We have already seen on pp. 151, 152, and 166 that for large values of L this equation will be approximately valid directly after the moment of contact.

If we assume that no external electromotive forces reside at the points of contact of the two systems, the following equation (5) will hold at these points:

$$L_1\frac{d\phi_1}{dn}=L\frac{d\phi}{dn}, \quad \text{......................(25)}$$

where the index 1 refers to the system $\mathfrak{S}$. At all other points on the surface of system $\mathfrak{S}_A$ we shall evidently have

$$\frac{d\phi}{dn}=0. \quad \text{.........................(26)}$$

If before the contact were made the value of ϕ happened to be the same, $\phi=$ const., at all points of system $\mathfrak{S}$ that are afterwards brought in contact with system $\mathfrak{S}_A$, $\phi=$ const. would then satisfy the above conditional equations (24)–(26) for system $\mathfrak{S}_A$. As we know from Riemann's theorem (cf. § 21) that there is only one solution for ϕ—we exclude those that differ from this solution by only an additative constant which would be the same in both systems—that satisfies these conditional equations, $\phi=$ const. will be the solution sought. A special case of such a system is that where there is only one point of contact between the two systems and $\phi=\phi_1$.

The constancy of ϕ in system $\mathfrak{S}$ can be brought about, as we have already seen on p. 154, etc., by assuming that the external electromotive forces have a potential ψ, and that they are so distributed through it that none of the points brought in contact with system $\mathfrak{S}$ are insulated, so to speak, from one another, as the points a and b of figure 14 on p. 154.

If ϕ has different values at the several points of contact between the two systems, the above solution $\phi=$const. for the system $\mathfrak{S}_A$ can of course no longer hold. Here electric disturbances or currents will be set up in system $\mathfrak{S}_A$ directly upon making contact. In order that ϕ may have different values on the surface of system $\mathfrak{S}$, it is only necessary either that the external electromotive forces have no potential or that the points of system $\mathfrak{S}$, that are brought in contact with system $\mathfrak{S}_A$, are entirely insulated from one another, as indicated in the annexed figure. This follows from the investigations of § 19.

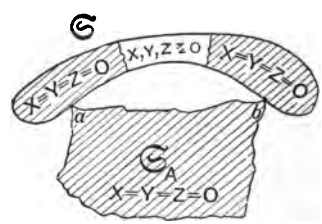

FIG. 16.

Let us next examine the following special case: A system $\mathfrak{S}$, similar to the above system $\mathfrak{S}$, is given; two points A and B of its surface are connected, the former with a second system of conductors, $\mathfrak{S}_A$, and the latter with a third system, $\mathfrak{S}_B$, within neither of which external electromotive forces reside. If the system $\mathfrak{S}_A$ or $\mathfrak{S}_B$ is otherwise, except in the points A or B respectively, completely insulated—these two systems shall also be insulated from each other—as indicated in the

annexed figure, the conditional equations for the system $\mathfrak{S}\mathfrak{S}_A\mathfrak{S}_B$ will evidently be fulfilled by assuming the value ϕ_A or ϕ_B for ϕ at any point in the interior of system $\mathfrak{S}_A$ or $\mathfrak{S}_B$ respectively, where ϕ_A or ϕ_B denotes the constant

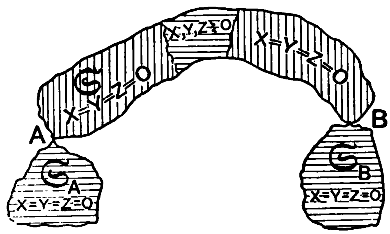

FIG. 17.

value of ϕ at the point A or B respectively. Here, as in the above cases, we can of course add an arbitrary constant to ϕ, which must, however, be the same in all three systems. We have seen above how such a difference of potential, $\phi_B - \phi_A$, can be produced on the surface of a conductor $\mathfrak{S}$, within which external electromotive forces reside. This difference will alone be determined by the form of the potential of the external electromotive forces, provided they have one (cf. formulae (25, VIII.)), and, if not, by their nature and position in the system $\mathfrak{S}$, that is, by the configuration of this system, and the position of the points A and B upon it; in both cases this difference will, however, be entirely independent of the configuration and constitution of the systems $\mathfrak{S}_A$ and $\mathfrak{S}_B$. These results follow also from the investigations of § 19.

The above includes the well-known case of two conductors connected with the terminals of a galvanic element and maintained at a given difference of potential. It is hardly necessary to add that the foregoing investigations still remain valid, when the contact between the two systems $\mathfrak{S}$ and $\mathfrak{S}_A$ or $\mathfrak{S}_B$ is made at several points, $A_1, A_2, \ldots$ or $B_1, B_2, \ldots$ respectively, provided the same potential, ϕ_A or ϕ_B respectively, prevailed at those points

before the contact. From § 19 it follows, moreover, that ϕ can, in general, only assume the same value in any series of points, when the external electromotive forces have a potential ψ.

To explain a difference of potential on the surface of contact of two conductors, we conceive, in conformity to the above, that a very thin conducting film of the same constitution as our system $\mathfrak{S}$ lies inserted between them. If such a film of external electromotive forces actually exists between two conductors, $\mathfrak{S}$ and $\mathfrak{S}_A$ or $\mathfrak{S}_B$, it can then be regarded as part of the former. These cases can, therefore, all be included under the above, which have already been investigated.

Equations (24)–(26) define analytically that problem, which is usually designated as the most general problem of electrostatics; it can be stated as follows: Any system of conductors is placed in any dielectric—the latter may also contain isolated quantities of real electricity, and any of the conductors, within which no external electromotive forces reside, are connected with any of those within which external electromotive forces that are independent of the time are active—the latter constitute our system of conductors $\mathfrak{S}$, but no two points of the system $\mathfrak{S}$, for which ϕ has different values, may be connected twice, that is, we exclude all connections between conductors, within which no external electromotive forces reside, that would give rise to a flow of electricity through them (cf. figure 16). We do not, however, exclude the case where any of the conductors of system $\mathfrak{S}$ are traversed by electric currents, whose origin is to be attributed to the fact that X, Y, Z have no potential, that is, that they are not the partial differentials with regard to the coordinates of a single-valued function (cf. p. 159). The total initial quantity of real electricity on the surface of any conductor that is brought in contact with the system $\mathfrak{S}$ must, of course, be known; this will evidently be equal to the total initial quantity of free electricity on its surface,

provided $D = \epsilon = 1$ in the surrounding dielectric. The difference of potential $\phi_B - \phi_A$ between any two points A and B on the surface of any conductor of the system $\mathfrak{S}$ is called the electromotive force acting between those points. The given problem includes the special cases, where conductors, within which external electromotive forces reside, are connected by thin conducting threads or wires, where no external forces are active, with an initially unelectrified conductor, that either completely envelopes the given conductors or is large in comparison to and at a great distance from them—these cases are indeed only special forms of the less general cases or systems 1 and 2 respectively on pp. 153-154 (cf. also text p. 159); here ϕ will evidently have the same value in the conductors and the initially unelectrified conductor, and, as this value is entirely arbitrary, it can thus be put equal to zero. Let a conductor connected with the earth by a wire serve as an illustration of such a system. If external electromotive forces reside in the conducting thread or wire, the value of ϕ in the conductors will, of course, differ from that of ϕ in the initially unelectrified conductor by a constant.

If the conditions that characterize the above general problem are not realized, the state of the ether will be that of stationary flow. The most general case of stationary flow, where, namely, entirely arbitrary external electromotive forces act in conductors of any constitution and configuration, has never been thoroughly investigated. A special case of this most general problem has already been briefly examined at the end of the last chapter. In the following we shall restrict ourselves to given special cases that are approximately realized. Such an one which is of special interest is that where two points A and B of a conductor $\mathfrak{S}$, that are maintained at different potentials, are connected with each other by another conductor (wire) $\mathfrak{S}'$, within which external electromotive forces are wanting (cf. figure 16). We can put the electrostatic potential at one of these

points, as A, equal to zero for both conductors, denote that at any point of the conductor $\mathfrak{S}$ before the contact by χ and that at its other point of contact B by $\chi_B = a$; a will then be the electromotive force acting between the points A and B (cf. p. 179). Let us denote by ψ' the function that is uniquely determined by the following conditions:

(1) $$\nabla^2 \psi' = 0,$$

or, if L is variable,

$$\frac{d}{dx}\left(L\frac{d\psi'}{dx}\right)+\frac{d}{dy}\left(L\frac{d\psi'}{dy}\right)+\frac{d}{dz}\left(L\frac{d\psi'}{dz}\right)=0$$

at every point in the interior of the conductor $\mathfrak{S}'$,

(2) $$\frac{d\psi'}{dn}=0$$

at every point on its surface, except A and B; and, lastly (3), $\psi'_A = 0$, and $\psi'_B = 1$. Let the function ψ be characterized by analogous conditions in the conductor $\mathfrak{S}$. After the contact has been made, the conditions (24) and (26) can evidently be satisfied by putting

$$\left.\begin{aligned} &\phi = \chi - \theta\psi \quad \text{for conductor } \mathfrak{S} \\ \text{and} \qquad &\phi = \theta'\psi' \quad \text{for conductor } \mathfrak{S}' \end{aligned}\right\}, \qquad (27)$$

where θ and θ' are arbitrary constants to be determined directly.

We know now that the total quantity of neutral electricity, not only the positive fluid that flows in the one direction but the negative that flows in the opposite direction, that passes any cross-section of the conductor $\mathfrak{S}$ or $\mathfrak{S}'$, must be the same for every cross-section, since the neutral electricity behaves like an incompressible fluid (cf. § 6). By formulae (6, III.), (14, V.) and (27) the quantity of neutral electricity that passes any cross-section of the conductor $\mathfrak{S}'$ per unit time will be

$$i = \int L\frac{d\phi}{dn}do = \theta'\int L\frac{d\psi'}{dn}do, \qquad (28)$$

where the normal n to the surface-element do of the given cross-section is to be drawn in the opposite direction to that of the flow. The value of this integral will evidently depend only on the constitution and configuration of the conductor $\mathfrak{S}'$ and the position of the points A and B on its surface, being entirely independent of the character of both the conductor $\mathfrak{S}$ and the external electromotive forces that reside within it. The reciprocal value of this integral is known as the resistance of the conductor ($\mathfrak{S}'$) between the points A and B; let us denote it by W'.

Similarly, we find the following expression for the flow of the neutral electricity through any cross-section of the conductor $\mathfrak{S}$:

$$i = -\int L\left(\frac{d\chi}{dn} - S\right)do + \theta\int L\frac{d\psi}{dn}do,$$

where S is the component of the vector (X, Y, Z) along the normal n to do.

According to the above, the direction of the positive normal n to do is from A towards B in both conductors, as indicated in the annexed figure. The direction of

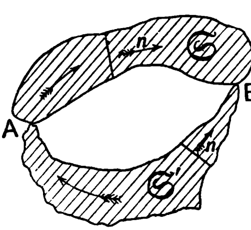

FIG. 18.

flow in the conductor $\mathfrak{S}$ will therefore be that of the normal n, and in the conductor $\mathfrak{S}'$ the opposite, as indicated.

The first integral of the last expression for i vanishes for every cross-section of the conductor $\mathfrak{S}$, since before the contact the ether was in the state of stationary flow,

and hence the resultant flow of the neutral electricity across any cross-section of it must have then been zero. For the conductor $\mathfrak{S}$ we shall have, therefore,

$$i=\theta\int L\frac{d\psi}{dn}do=\frac{\theta}{W}, \quad\text{.................(29)}$$

where W denotes the resistance of the conductor $\mathfrak{S}$.

Both of the above solutions (27) for ϕ must, of course, hold at the point of contact B that is common to both conductors $\mathfrak{S}$ and $\mathfrak{S}'$. At this point

$$\begin{gathered}\psi=\psi'=1, \quad \chi=a, \\ \phi_B=a-\theta=\theta', \quad\text{.........................(30)}\end{gathered}$$

from which we can determine the constants θ and θ'. We see, therefore, that θ' is the electromotive force acting between the two points A and B. Any difference of potential that might be produced at the points of contact of the two conductors or metals is to be included under the potential-differences arising from the external electromotive forces of the conductor $\mathfrak{S}$.

That the function ϕ may be uniquely determined, it is only necessary that the conditions (25) which have been omitted above, should be fulfilled; for the given system these conditions are evidently

$$\left.\begin{aligned}&\left(L\frac{d\chi}{dn}\right)_A-\theta\left(L\frac{d\psi}{dn}\right)_A=\theta'\left(L\frac{d\psi'}{dn}\right)_A\\ \text{and}\quad&\left(L\frac{d\chi}{dn}\right)_B-\theta\left(L\frac{d\psi}{dn}\right)_B=\theta'\left(L\frac{d\psi'}{dn}\right)_B\end{aligned}\right\}, \quad\text{.........(31)}$$

where the index A or B denotes the value of the given quantity at the point A or B respectively. Here the quantities

$$\left(L\frac{d\chi}{dn}\right)_A,\ \left(L\frac{d\chi}{dn}\right)_B,\ \left(L\frac{d\psi'}{dn}\right)_A, \text{ and } \left(L\frac{d\psi'}{dn}\right)_B,$$

are to be regarded as known. By replacing the constants θ and θ' by their values (30) in conditions (31) we find that

the following relations must hold between the functions ψ and ψ', in order that ϕ may be uniquely determined:

$$\left.\begin{aligned} &\left(L\frac{d\chi}{dn}\right)_A-(a-\phi_B)\left(L\frac{d\psi}{dn}\right)_A=\phi_B\left(L\frac{d\psi'}{dn}\right)_A \\ \text{and}\quad &\left(L\frac{d\chi}{dn}\right)_B-(a-\phi_B)\left(L\frac{d\psi}{dn}\right)_B=\phi_B\left(L\frac{d\psi'}{dn}\right)_B \end{aligned}\right\}\quad\ldots\ldots(32)$$

These conditions can now always be fulfilled by giving the quantities $\left(L\frac{d\psi}{dn}\right)_A$ and $\left(L\frac{d\psi}{dn}\right)_B$ the values determined from the conditions themselves; hereby the function ψ becomes uniquely determined, for not only the conditions (24) and (26) but also the conditions (25), all of which are necessary for the unique determination of any function ψ, will then be fulfilled. The further conditions for ψ, $\psi_A=\psi'_A=0$ and $\psi_B=\psi'_B=1$, can always be satisfied by a suitable choice of the zero-point and the system of units employed for the measurement of ψ and ψ'. For an actual determination of such a function ψ', see below.

Formulae (28)–(30) give the relation

$$i=\frac{a-\phi_B}{W}=\frac{\phi_B}{W'},\quad\ldots\ldots\ldots\ldots\ldots\ldots(33)$$

which is known as Ohm's law. Its present derivation may perhaps seem unnecessarily complicated; its value will, however, become apparent upon examining the behaviour of electromotive forces in other than linear conductors. We observe here that Ohm's law was established in its original form for linear conductors only; its validity for conductors of any configuration was proved later by Kirchhoff.*

* "Ueber die Anwendbarkeit der Formeln für die Intensitaten der galvanischen Ströme in einem Systeme linearer Leiter auf Systeme, die zum Teil aus nicht linearen Leitern bestehen." Poggendorf's *Annalen* v. 75, 1848. *Gesammelte Abhandlungen*, pp. 33-49.

As illustration of the above formulae let us examine the action of electromotive forces in a linear homogeneous conductor or wire of length λ and cross-section q. Let the axis of the conductor be chosen as x-axis and the point A as origin of our system of coordinates. We know then that at any point of the conductor

$$\nabla^2\phi' = \frac{d^2\phi'}{dx^2} = 0, \quad (L = \text{const.}),$$

which integrated gives

$$\phi' = ax + \beta, \quad \phi'_A = \beta = 0 \quad \phi'_B = a\lambda,$$

hence

$$\phi' = \frac{\phi'_B}{\lambda} x.$$

Since by formula (27)

$$\phi' = \theta'\psi'$$

and by formula (30)

$$\phi'_B = \theta',$$

it follows that

$$\psi' = \frac{x}{\lambda}.$$

This function $\psi' = \frac{x}{\lambda}$ will also satisfy condition (26), namely

$$\frac{d\psi'}{dn} = \frac{d\psi'}{d(\sqrt{y^2 + z^2})} = 0,$$

which must hold at every point except A and B on the surface of the conductor. The only remaining conditions that ϕ must satisfy in order that it may be uniquely determined are the conditions (25), which assume here the special form

$$\left(L\frac{d\chi}{dn}\right)_A - \left(a - \phi_B\right)\left(L\frac{d\psi}{dn}\right)_A = \frac{\phi_B L}{\lambda}$$

and

$$\left(L\frac{d\chi}{dn}\right)_B - \left(a - \phi_B\right)\left(L\frac{d\psi}{dn}\right)_B = \frac{\phi_B L}{\lambda};$$

these conditions can evidently always be fulfilled and the function ψ thus uniquely determined (see above).

The current-strength i will be given by formulae (28) etc., namely

$$i = \phi_B L \int \frac{d\psi'}{dx} do = \phi_B L \int \frac{do}{\lambda}, \ (L = \text{const.});$$

the resistance W' of the given conductor will thus be

$$W' = \frac{\lambda}{Lq}, \text{ and hence } i = \frac{\phi_B}{W'}.$$

Kirchhoff's formulae for the distribution of electric currents in any network of conductors or wires arranged in multiple arc can be obtained directly from our general relations (33); we observe that their validity is not restricted to linear conductors, but is only subject to the condition that the contacts between the given conductors are made in single points. The principle of the Wheatstone bridge will, of course, also follow directly from these formulae.

We have already often remarked that the neutral electricity behaves like an incompressible fluid; consequently, the sum of all the electricities that enter any point, in which two or more conductors meet, will always be equal to the sum of all the electricities that escape from it, that is, the algebraic sum of all the electricities entering any point will vanish and we shall thus have

$$i_1 + i_2 + i_3 + \ldots + i_n = 0,$$

where n denotes the number of conductors that meet at the given point. This formula is known as Kirchhoff's first law.

Let us next examine any complete circuit of conductors $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \ldots \mathfrak{S}_n$, with the single points of contact $A_{1,2}, A_{2,3}, \ldots A_{n-1,n}, A_{n,1}$; let $A_{k-1,k}$ denote the point of contact between any two adjacent conductors $\mathfrak{S}_{k-1}$ and $\mathfrak{S}_k$. Let the function ψ_k refer to the conductor

$\mathfrak{S}_k$ and be determined by a differential equation and surface-conditions similar to those that define the functions ψ and ψ' above, the value of ψ_k at the point $A_{k-1,k}$ being 0 and at the point $A_{k,k+1}$ unity; let the value of ϕ at the former point be $\theta_{k-1,k}$. Lastly, let

$$\int L\frac{d\psi_k}{dn}do=\frac{1}{W_k}.$$

The function

$$\phi_k=(\theta_{k,k+1}-\theta_{k-1,k})\psi_k+\theta_{k-1,k}$$

will then satisfy the differential equation (24) and the surface-condition (26)—the surface-conditions (25) which are similar to those for ϕ on p. 182 are of no further

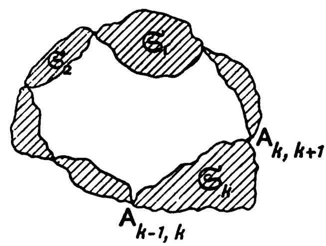

FIG. 19.

interest; it also has the desired values at the points of contact $A_{k-1,k}$ and $A_{k,k+1}$. Here we have, however, excluded external electromotive forces from the given conductor. If external forces reside in any conductor $\mathfrak{S}_k$, let the value of their potential at any point before the contact be denoted by χ_k; moreover, let $\chi_k=0$ at the point $A_{k-1,k}$ and $\chi_k=a_k$ at the point $A_{k,k+1}$. In this case ϕ_k must evidently have the form

$$\phi_k=\theta_{k-1,k}+(\theta_{k,k+1}-\theta_{k-1,k}-a_k)\psi_k+\chi_k. \quad\text{......(34)}$$

This value will satisfy all conditions for ϕ_k, provided the two arbitrary constants are given the values

$$\theta_{k-1,\,k} = (\phi_k)_{k-1,\,k},$$
$$\theta_{k,\,k+1} = (\phi_k)_{k,\,k+1};$$

their difference, $\theta_{k,\,k+1} - \theta_{k-1,\,k} = (\phi_k)_{k,\,k+1} - (\phi_k)_{k-1,\,k}$, is the electromotive force acting between the two given points of contact.

From the above value for ϕ_k it follows that

$$i_k = \int L_k\left(\frac{d\phi_k}{dn} - S_k\right)do$$
$$= (a_k - \theta_{k,\,k+1} + \theta_{k-1,\,k})\int L_k\frac{d\psi}{dn}do - \int L_k\left(\frac{d\chi_k}{dn} - S_k\right)do$$

or, since
$$\int L_k\left(\frac{d\chi_k}{dn} - S_k\right)do = 0,$$

$$i_k = (a_k - \theta_{k,\,k+1} + \theta_{k-1,\,k})\int L\frac{d\psi_k}{dn}do$$
$$= (a_k - \theta_{k,\,k+1} + \theta_{k-1,\,k})\frac{1}{W_k}.$$

As this equation holds for every conductor of the given circuit, we shall thus have

$$i_1 = (a_1 - \theta_{1,\,2} + \theta_{0,\,1})\frac{1}{W_1},$$
$$i_2 = (a_2 - \theta_{2,\,3} + \theta_{1,\,2})\frac{1}{W_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$i_n = (a_n - \theta_{0,\,1} + \theta_{n-1,\,n})\frac{1}{W_n}.$$

Adding these equations, we find the simple relation

$$\Sigma i_k W_k = \Sigma a_k,$$

which is known as Kirchhoff's second law.

If the given circuit is composed of linear homogeneous conductors or wires, the function ψ can be determined as

on p. 184 for every conductor separately, and hence its resistance and current-strength also. For a more dilate treatment of the distribution of electric currents in given systems of linear conductors we refer the student to §§ 273-284 of Maxwell's *Treatise on Electricity and Magnetism.*

If in any system of conductors the constant of conduction L of any given conductor is small in comparison to that of those surrounding it, it follows from equation (25) that $\frac{d\phi}{dn}$ will approximately vanish at all points in the latter close to the surface of the former. The given conductor will, therefore, behave like a non-conductor. It thus follows that a system of good conductors surrounded by bad ones will behave for a considerable period as if it were insulated. If, on the other hand, the constant of conduction L of any single conductor, within which no external electromotive forces reside, is large compared to that of the surrounding media, $\frac{d\phi}{dn}$ will vanish on the surface of the given conductor—the normal n is to be drawn into the conductor. We shall have then at any point of its interior

$$\frac{d}{dx}\left(L\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(L\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(L\frac{d\phi}{dz}\right)=0\,;$$

or, if L is constant, $\nabla^2\phi=0,$

from which it follows that ϕ will be constant throughout the conductor.

CHAPTER X.

SECTION XXIII. ANALOGY BETWEEN ELECTROSTATICS AND THE THEORY OF STATIONARY FLOW; EXAMPLES AS ILLUSTRATIONS.

IN the theory of the stationary flow of electricity over surfaces and through bodies, it is customary to assume that the external electromotive forces do not reside in the surfaces or bodies themselves, but in bodies whose constant L of conduction is very large, and which are connected with the former at two points called the electrodes; ϕ will then have a given value at each electrode. The following two problems, whose solutions are analogous, then present themselves. The first can be stated as follows: Two conductors, entirely insulated from each other, and within which no external electromotive forces reside, are placed in a dielectric or system of dielectrics, from which all real electricity is excluded, and the state of the ether is sought at any point of the system. It follows, then, from the considerations on the preceding page that ϕ will have a given constant value in each conductor or layer of the given condenser. At any point in any of the surrounding dielectrics

$$\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\frac{d}{dy}\left(D\frac{d\phi}{dy}\right)+\frac{d}{dz}\left(D\frac{d\phi}{dz}\right)=0, \quad\ldots\ldots\ldots(1)$$

and on the dividing-surface between any two dielectrics

$$D_1\frac{d\phi_1}{dn}=D_0\frac{d\phi_0}{dn}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The total quantity of real electricity on either layer of the given condenser will be

$$R=\int E_r do = -\frac{1}{4\pi}\int D\frac{d\phi}{dn}do \quad\text{.................(3)}$$

(cf. formula (4, VII.)), where the value of $\frac{d\phi}{dn}$ is to be taken in the dielectric close to the dividing-surface-element do. The second problem differs from the first in only two respects, namely, instead of the above dielectrics we have a system of conductors of different specific conduction L, and, instead of the above conductors, conductors within which external electromotive forces reside. The same equations hold for this system as for the former, provided we only write $4\pi L$ instead of D in the latter. Here the quantity

$$I=-\int L\frac{d\phi}{dn}do \quad\text{........................(4)}$$

corresponds to the quantity R of the first problem; it represents the quantity of neutral electricity that enters the system at the given electrode per unit-time, that is, the intensity of the current or the current-strength at that point; the normal n is to be drawn here from the electrode into the conductor.

If the value of I at both electrodes is the same, but opposite in sign, no electricity will escape from the system, whereas, if it is different, the algebraic sum of these two quantities of electricity will be dissipated into space. It is only the former case, that of stationary flow, that interests us here; one of its most common forms is that where the one electrode completely surrounds the other. If the one layer of a condenser or the one electrode is enclosed by the other, it follows, therefore, from the above analogy that the algebraic sum of the electricities on both its surfaces will also be equal to zero. This follows, moreover, from the condition that

$$\nabla^2\phi=0,$$

and the equation

$$\int \nabla^2 \phi d\tau = \int \frac{d\phi}{dn} do,$$

obtained from the following form of Green's law by putting $U=1$, $V=\phi$:

$$\int (U\nabla^2 V - V\nabla^2 U) d\tau = \int \left(U\frac{dV}{dn} - V\frac{dU}{dn} \right) do.$$

If the algebraic sum of the electricities on both layers of a condenser vanishes, we call R/b, where b denotes the difference of potential between its two layers, its capacity. The capacity of a single conductor can be similarly defined by conceiving that the wanting layer is replaced by a good conducting spherical shell of infinite radius, enclosing the given conductor and maintained at the same constant potential as the earth. Analogously, when the value of I at the two electrodes is the same but opposite in sign, the quotient I/b is called the reciprocal resistance ($1/W$) of the system (conductor).

We should not, however, fail to observe a quantitative difference between the two problems. A form of the latter system, that is almost exclusively met with in our laboratories, is that where only certain regions of space are conducting, in fact, where the given conductors have the form of thin, good-conducting shells or sheets or even threads (wires), and all the remaining regions are non-conducting; at any point on the surface of such a conductor $\frac{d\phi}{dn}$ will be approximately equal to zero (cf. p. 188). On the other hand, the analogous system in the first problem, where, namely, the constant of electric polarization (induction) D of certain dielectrics is large in comparison to that of the surrounding medium, is one that is perhaps nowhere realized in nature.

Let us observe here, preliminarily, that the equations for magnetic induction are exactly similar to those for

electric induction (polarization), and hence, in conformity to the above analogy between electrostatics and stationary flow, that the laws of magnetic induction will be analogous to those of stationary flow. The constant of magnetic induction M of certain bodies, as iron, is now very large compared with that of other bodies, as air; it follows, therefore, that masses of iron, upon being magnetized by induction, may be regarded approximately as (magnetic) conductors surrounded by an insulating medium. On account of the analogous rôles played by these three medium-constants, D, L, and M, it has thus become customary to speak of the conduction of electric induction from the one layer of a condenser to the other, and to designate D as the constant of conduction of the electric induction (polarization) or its conductivity, expressions that have already been universally accepted in the theory of magnetic induction.

In dielectrics it is customary to imagine a system of curves that coincide at every point with the direction of the vector N whose components are P, Q, R. In order to determine the density of these curves at any point we lay a surface-element do at right angles to their direction. The number of curves that pass through this element divided by its area will then be equal to the number of curves that pass through unit-surface in the direction of the normal. Let this quotient always be chosen equal to the value of DN at the given point. These curves are known as the lines of force, or, better, as the lines of electric induction (polarization). The components DP, DQ, and DR of the vector DN correspond to the components of the current LP, LQ, and LR in a conductor, and the direction and strength of the electric polarization to the direction and strength of the current. To make these quantities correspond to one another numerically, we should have to choose $\frac{DN}{4\pi}$ instead of DN as the number of lines of force that

pass through unit-surface in the direction of the normal. Maxwell does this, in fact, and writes $f=\frac{DP}{4\pi}$, ... and $a=Ma$, ... On account of this similarity between the lines of flow and those of electric polarization, it is customary to speak of the conduction of the latter.

Let us determine the number of lines of electric induction that enter and leave respectively any volume-element $dxdydz$ of space; the number of lines that enter it through its one side $dydz$ will then be

$$dydz\,.\,DP,$$

and the number that leave its opposite side

$$dydz\,DP(x+dx,\,y,\,z)=dydz\,DP+dxdydz\frac{d(DP)}{dx}.$$

Similar expressions will hold for its other two pairs of sides. The excess of the number of lines that leave the volume-element over that of those that enter it, that is, the number of lines of force that are created within it, will then be

$$dxdydz\left[\frac{d}{dx}(DP)+\frac{d}{dy}(DQ)+\frac{d}{dz}(DR)\right];$$

if this expression is negative, lines will end instead of begin in the parallelopiped.

Comparing this expression with that (1, VI.) for the density of the real electricity at any point of a dielectric, we observe that wherever real electricity resides lines of electric polarization either begin or end, whereas, if this expression vanishes, no electricity can appear. Similarly, we can prove that, if electricity flows into or escapes from any point of a conductor, lines of flow must begin or end at that point.

Let us next examine the following special cases:

(1) Let ϕ be a function of x only; for $x=0$ let $\phi=0$, and for $x=a$, $\phi=b$. $\nabla^2\phi$ then reduces to

$$\frac{d^2\phi}{dx^2}=0,$$

which, integrated, gives $\phi=\frac{b}{a}x$.

In the first problem the planes $x=0$ and $x=a$ correspond to the two layers of a condenser. By formula (3) the quantity of real electricity on the surface Q of one of these layers will be

$$[R]=\frac{D}{4\pi}\frac{b}{a}Q,$$

where the brackets denote the absolute value of the given quantity. The capacity C of the condenser will therefore be

$$C=\frac{DQ}{4\pi a};$$

if the intervening medium were the real standard medium, $D=1$, we should have

$$C_a=\frac{Q}{4\pi a}. \qquad (5)$$

The quantity D (cf. formula (3, II.)) can, therefore, be defined as the quotient of these two capacities, C/C_a, and thus be determined experimentally.

If the constant of electric induction of the dielectric between the two layers of the condenser is much larger than that of the surrounding medium, the above formulae will still remain approximately valid, even when the distance between the two layers is increased to the dimensions of their surface-areas Q.

Analogous equations hold for the second problem. Here two very good conducting plates of surface-area Q replace the two layers of the condenser and a conductor

that of the intervening dielectric. Instead of the above expression for R we have here the corresponding expression

$$I = L\frac{b}{a}Q$$

for the value of the current-strength, and, instead of the above expression for the capacity C, the following for the reciprocal resistance:

$$\frac{1}{W} = \frac{I}{b} = \frac{LQ}{a}.$$

The specific conductivity L can thus be defined as the quantity of electricity that flows through unit-cross-section, when the potential difference per unit-length is unity; it is measured here in the electrostatic system of units; if any other system of units were employed, L would have to be replaced by L_h (cf. formula (33, IX.)) in order to avoid the appearance of a constant factor in Ohm's law.

In this second problem the specific conductivity of the surrounding medium is often so poor that the above conditions still hold, when Q is small, in fact, even when the conductor, whose resistance is sought, has the form of a very fine wire (cf. also p. 191).

(2) A system of special interest is the electrostatic system, that differs from the above in only the one respect, that, namely, a second dielectric is inserted in the medium, which separates the two layers of the condenser. Suppose that the inserted dielectric is in the form of a stratum parallel to the layers of the given condenser; let a pane of glass placed between the layers of a condenser surrounded by air serve as an illustration of such a system. We designate the region between the inserted dielectric and the left or right layer of the condenser as the region I. or III. respectively, suffixing the index 1 or 3 to all quantities that refer to that region, and the inserted dielectric (the pane of glass)

as the region II., with the corresponding index 2 suffixed to all its quantities. Let the thicknesses of these regions be a_1, a_2, and a_3 respectively; $a_1+a_2+a_3=a$ is then the distance between the two layers of the condenser. We have

$$\phi_1=Ax+B, \quad \phi_2=A'x+B', \quad \phi_3=A''x+B''. \quad \text{.....(6)}$$

From the conditions that for $x=0$, $\phi=0$, and for $x=a$, $\phi=b$, it follows then that

$$B=0, \text{ and } B''=b-aA.$$

For $x=a_1$ we have

$$[\phi_1]_{a_1}=a_1A$$

and

$$[\phi_2]_{a_1}=a_1A'+B',$$

and for $x=a_1+a_2$ $\quad [\phi_2]_{a_1+a_2}=(a_1+a_2)A'+B'$

and

$$[\phi_3]_{a_1+a_2}=(a_1+a_2)A''+B'',$$

which, since

$$[\phi_1]_{a_1}=[\phi_2]_{a_1}, \text{ and } [\phi_2]_{a_1+a_2}=[\phi_3]_{a_1+a_2},$$

give

$$a_1A=a_1A'+B', \text{ and } (a_1+a_2)A'+B'=(a_1+a_2)A''+B''.$$

The following condition (cf. formula (27, VI.)) must now hold on the dividing-surface between any two dielectrics:

$$D_1\frac{d\phi_1}{dn}=D_0\frac{d\phi_0}{dn}.$$

For the surface $x=a_1$ this condition reduces to the following after replacing ϕ by its values (6):

$$D_1A=D_2A',$$

and for the surface $x=a_1+a_2$ to

$$D_2A'=D_3A''.$$

Since $D_1 = D_3 = D$, these last two conditions give

$$A' = \frac{D}{D_2} A = \frac{D}{D_2} A'',$$

hence

$$A = A''.$$

The above conditional relations suffice for the determination of the six arbitrary constants, A, A', A'', B, B', B''. We find the following values:

$$A = A'' = \frac{\lambda b}{\lambda(a - a_2) + a_2}, \quad A' = \frac{b}{\lambda(a - a_2) + a_2},$$

$$B = 0, \quad B' = \frac{(\lambda - 1) a_1 b}{\lambda(a - a_2) + a_2}, \quad B'' = \frac{(1 - \lambda) a_2 b}{\lambda(a - a_2) + a_2},$$

where $\lambda = \frac{D_2}{D}$.

We thus obtain the following values for ϕ:

$$\phi_1 = \frac{\lambda b x}{\lambda(a - a_2) + a_2}, \quad \phi_2 = \frac{b x + (\lambda - 1) a_1 b}{\lambda(a - a_2) + a_2},$$

$$\phi_3 = \frac{\lambda b x + (1 - \lambda) a_2 b}{\lambda(a - a_2) + a_2}.$$

By formula (3) the total quantity of electricity on either layer of the condenser will therefore be

$$[E] = \frac{1}{4\pi} \int D \frac{\lambda b \, do}{\lambda(a - a_2) + a_2} = \frac{D_2}{4\pi} \cdot \frac{b Q}{\lambda(a - a_2) + a_2},$$

where Q denotes the surface-area of either layer. The capacity of the condenser will thus be

$$\frac{D_2 Q}{4\pi\lambda\left(a + a_2 + \frac{a_2}{\lambda}\right)}.$$

Observe that this expression is entirely independent of the quantities a_1 and a_3, depending only on a_2, the thickness of the inserted dielectric, and a, the

distance between the layers of the condenser; that is, it is entirely independent of the relative position of the inserted dielectric. If we choose the intervening dielectric as our real standard medium, putting $D=1$, and denote the constant of electric induction of the inserted dielectric measured in this standard system of units by D, we can then write the above expression for the capacity of the condenser in the form

$$\frac{Q}{4\pi\left(a-a_2+\frac{a_2}{D}\right)};$$

it thus follows that the insertion of a second dielectric, as a pane of glass, between the layers of a condenser alters its capacity, for $\left(a-a_2+\frac{a_2}{D}\right)$ now takes the place of a (cf. formula (5)). If D is larger than unity [as for glass ($D=2{\cdot}6$) or paraffin ($D=2{\cdot}32$)], the capacity of the condenser is increased by the insertion of such a second dielectric by the factor

$$\frac{\dfrac{1}{a-a_2+\frac{a_2}{D}}}{\dfrac{1}{a}}=\frac{a}{a-a_2+\frac{a_2}{D}}.$$

The analogous problem in the theory of stationary flow is hardly realized in nature, since external electromotive forces generally reside in the surfaces of contact of conductors or metals.

We should observe that the two preceding problems can only be approximately realized; for, in order that ϕ may be a function of x only, the electrodes must be infinitely large, that is $Q=\infty$, since ϕ would become a function of y and z as we approached the edges of the electrodes. In determining the capacity C of a condenser or the resistance W of a conductor of finite dimensions,

we should therefore have to take into consideration this variation in ϕ near the edges of the electrodes. A method for the exacter determination of the capacity of such a condenser has been given by Clausius;* later, a much simpler method was found by Kirchhoff.† This small correction in C or W, due to this variation in ϕ, is, however, eliminated partly in the next two problems, (3) and (4), and entirely in the 5th and 6th.

(3) The two electrodes are coaxial cylindrical surfaces of radii r_0 and r_1 respectively. ϕ can then be regarded as a function of the radius-vector r only, and our differential equation $\nabla^2\phi$ thus written in the familiar form,

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0,$$

whose integral is

$$\phi = a \log r + A,$$

where a and A are the two arbitrary constants. If ϕ_0 and ϕ_1 denote the values of the potential on the inner and outer cylinders respectively, the following conditional equations must hold:

$$\phi_0 = a \log r_0 + A, \quad \phi_1 = a \log r_1 + A\,;$$

these give the following values for the constants a and A:

$$a = \frac{\phi_1 - \phi_0}{\log r_1 - \log r_0}, \quad A = \phi_0 - \frac{(\phi_1 - \phi_0)\log r_0}{\log r_1 - \log r_0}.$$

Hence $$\phi = \frac{\phi_1 - \phi_0}{\log r_1 - \log r_1} \log r + \phi_0 - \frac{(\phi_1 - \phi_0)\log r_0}{\log r_1 - \log r_0}.$$

The total quantity of real electricity on and the total

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* Poggendorf's *Annalen*, v. 86.

† *Zur Theorie des Condensators; Monatsbericht der Akademie der Wissenschaften zu Berlin*, v. 15, *März*, 1877; and *Gesammelte Abhandlungen*, pp. 101-117.

current-strength at either electrode will, therefore, be

$$\frac{DQ}{4\pi}\cdot\frac{\phi_1-\phi_0}{\log r_1-\log r_0}=\frac{lD(\phi_1-\phi_0)}{2(\log r_1-\log r_0)}$$

and $$LQ\frac{\phi_1-\phi_0}{\log r_1-\log r_0}=\frac{2\pi lL(\phi_1-\phi_0)}{\log r_1-\log r_0}$$

respectively, where l denotes the length of either cylinder.

Hence the capacity of the condenser and the reciprocal resistance of the conductor will be

$$\frac{lD}{2(\log r_1-\log r_0)} \text{ and } \frac{2\pi lL}{\log r_1-\log r_0} \text{ respectively.}$$

(4) A system consisting of the two coaxial cylindrical surfaces of problem (3), and a coaxial cylindrical stratum —a second dielectric or conductor, as the case requires— inserted in the intervening medium between the two electrodes, may serve as another illustration of the analogy between electrostatics and the theory of stationary flow. Its treatment is similar to that of problem (2), whereas the equations, with which one has to operate, are those of the preceding problem.

(5) For two concentric spherical shells of radii r_0 and r_1, our conditional equation $\nabla^2\phi=0$ reduces to the familiar form

$$\nabla^2\phi=\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}=0,$$

whose integral is

$$\phi=\frac{a}{r}+A.$$

We determine the arbitrary constants a and A in the usual manner and we find

$$a=\frac{\phi_1-\phi_0}{\frac{1}{r_1}-\frac{1}{r_0}}, \qquad A=\phi_0-\frac{\phi_1-\phi_0}{\frac{r_0}{r_1}-1},$$

and hence

$$\phi = \frac{\phi_1 - \phi_0}{\frac{1}{r_1} - \frac{1}{r_0}} \frac{1}{r} + \phi_0 - \frac{\phi_1 - \phi_0}{\frac{r_0}{r_1} - 1}.$$

The total quantity of electricity on either spherical shell and the total current-strength through the conductor between the two shells will therefore be

$$\frac{DQ}{4\pi} \cdot \frac{\phi_1 - \phi_0}{\frac{1}{r_1} - \frac{1}{r_0}} \frac{1}{r^2} \text{ and } LQ \frac{\phi_1 - \phi_0}{\frac{1}{r_1} - \frac{1}{r_0}} \frac{1}{r^2} \text{ respectively,}$$

or, since $Q = 4\pi r^2$,

$$D \frac{\phi_1 - \phi_0}{\frac{1}{r_1} - \frac{1}{r_0}} \text{ and } 4\pi L \frac{\phi_1 - \phi_0}{\frac{1}{r_1} - \frac{1}{r_0}} \text{ respectively,}$$

and hence the capacity of the former and reciprocal resistance of the latter

$$\frac{D}{\frac{1}{r_1} - \frac{1}{r_0}} \text{ and } \frac{4\pi L}{\frac{1}{r_1} - \frac{1}{r_0}} \text{ respectively.}$$

(6) A system similar to that of problem (4), but with concentric spherical surfaces in place of the coaxial cylindrical ones, may serve as still another illustration of the above analogy.

SECTION XXIV. ELECTRIC IMAGES.

We obtain somewhat more general formulae than those of the preceding article as follows: let r and r' denote the distances of any point P in any plane $z = \text{const.}$ from the two points A and B respectively in this plane, whose coordinates are $x = c$, $y = 0$, ($z = \text{const.}$), and $x = -c$, $y = 0$, ($z = \text{const.}$) respectively (cf. annexed figure). From

the figure we have the two following relations between the bipolar coordinates r and r' and the rectilinear coordinates x and y:

$$r^2=(x-c)^2+y^2, \quad r'^2=(x+c)^2+y^2. \qquad\ldots\ldots\ldots\ldots(7)$$

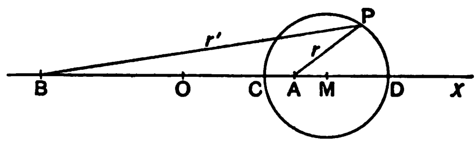

FIG. 20.

The following integral satisfies here our differential equation $\nabla^2\phi=0$:

$$\phi=g\log\frac{r'}{r}+g', \qquad\ldots\ldots\ldots\ldots(8)$$

where g and g' are the two arbitrary constants. This follows from the following value for ϕ, obtained from relations (7):

$$\phi=\tfrac{1}{2}g\log[(x+c)^2+y^2]-\tfrac{1}{2}g\log[(x-c)^2+y^2]+g'$$

and the following expressions:

$$\frac{d\phi}{dx}=g\left(\frac{x+c}{r'^2}-\frac{x-c}{r^2}\right), \quad \frac{d\phi}{dy}=gy\left(\frac{1}{r'^2}-\frac{1}{r^2}\right),$$

$$\frac{d^2\phi}{dx^2}=g/r'^2-g/r^2-\frac{2g(x+c)^2}{r'^4}+\frac{2g(x-c)^2}{r^4},$$

$$\frac{d^2\phi}{dy^2}=g/r'^2-g/r^2-\frac{2gy^2}{r'^4}+\frac{2gy^2}{r^4},$$

$$\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right)=2g\left[\frac{1}{r'^2}-\frac{1}{r^2}-\frac{(x+c)^2+y^2}{r'^4}+\frac{(x-c)^2+y^2}{r^4}\right],$$

which give

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0,$$

or since

$$\frac{d^2\phi}{dz^2}=0,$$

$$\nabla^2\phi=0.$$

The equipotential surfaces of the above solution (8) are now given by the equation

$$g\log\frac{r'}{r}+g'=\phi=\text{const.},$$

which can be written

$$r=ar', \quad\text{.........................(9)}$$

where a is a constant.

In rectilinear coordinates this equation assumes the form

$$(x-c)^2+y^2=a^2(x+c)^2+a^2y^2,$$

or

$$\left(x-\frac{c(1+a^2)}{1-a^2}\right)^2+y^2=\left(\frac{2ac}{1-a^2}\right)^2, \quad\text{...........(10)}$$

which we recognize as the equation of a circular cylindrical surface, whose radius is

$$\frac{2ac}{1-a^2},$$

and whose axis M is given by the intersection of the two planes

$$x=\frac{c(1+a^2)}{1-a^2} \quad\text{and}\quad y=0.$$

By assuming different values for a, we obtain a system of eccentric circular cylindrical equipotential surfaces.

Denoting the straight lines, in which the equipotential surface (10) intersects the plane $y=0$ by C and D (cf. figure 20), we find

$$\overline{OC}=c\frac{1-a}{1+a},\quad \overline{OD}=c\frac{1+a}{1-a}, \quad\text{..............(11)}$$

from which and the above figure it follows that

$$\overline{OC}\,.\,\overline{OD}=\overline{OA}^2,\quad \overline{AM}\,.\,\overline{BM}=\overline{CM}^2. \quad\text{..........(12)}$$

The above equations and analytic relations are the key to the ensuing important electrical problems.

Let us first examine the case, where a closed cylindrical surface (electrode) maintained at a constant potential encircles each axis A and B. Let the mean radius-vector of each surface from its axis A or B respectively be infinitely small and be denoted by a or β respectively. As $\nabla^2\phi$ vanishes here at every point of space (cf. p. 190), the above solution (8) for ϕ, including the resulting relations (11) and (12), will evidently be the one sought. At any point on the electrode A, whose radius-vector is a, we have

$$r=a, \text{ and } r'=2c \text{ approximately,}$$

and similarly for B,

$$r=2c \text{ approximately and } r'=\beta.$$

Hence $$\phi_a=g\log\frac{2c}{a}+g' \text{ and } \phi_\beta=g\log\frac{\beta}{2c}+g',$$

where ϕ_a and ϕ_β denote the mean values of the potential at the electrodes A and B respectively. The introduction of the given surfaces in place of the axes A and B was necessary, as is now apparent, in order to avoid the appearance of ∞ in our formulae (cf. also p. 218).

The above conditions give

$$g=\frac{\phi_\beta-\phi_a}{\log\frac{a\beta}{4c^2}};$$

ϕ thus assumes the form

$$\phi=\frac{\phi_\beta-\phi_a}{\log\frac{a\beta}{4c^2}}\log\frac{r'}{r}+g'.$$

As the arbitrary constant g' cannot appear in our final expressions for I, R, W, and C, since only the derivatives of ϕ come into consideration, its determination has been omitted here and in the following.

We have just seen that the equipotential surfaces for ϕ are circular cylinders, that is, conversely, that ϕ is constant on any one of these surfaces. Let us next suppose that the axis A is encircled by a circular cylindrical electrode of finite radius a, instead of by the above electrode, and let us seek the solution for ϕ. Observe here that the constitution of the region enclosed by this electrode has nothing whatever to do with the given problem; it may, therefore, be either of the same constitution as the surrounding electrode or a mere vacuum. It is evident that the solution of this problem can be found by means of the relations (11) and (12) of the preceding problem, namely, by choosing the axis B of the latter in such a manner that

$$\overline{AM}\,.\,\overline{BM}=\overline{CM}^2,$$

for the given electrode then becomes one of the equipotential surfaces of the resulting system, whose solution is already known. The desired solution for ϕ will therefore be

$$\phi=g\log\frac{r'}{r}+g',$$

where r' denotes the radius-vector of the given point from the axis B, which lies in the plane $y=0$ at the distance $\dfrac{\overline{CM}^2}{\overline{AM}}$ from the axis of the given electrode.

To determine the constant g we have the two conditional relations

$$\phi_a=g\log\frac{\overline{BC}}{\overline{AC}}+g'$$

(cf. figure 20), where ϕ_a denotes the given (constant) value of ϕ at any point of the given electrode, and

$$\phi_\beta=g\log\frac{\beta}{2c}+g',$$

as above; from which we find the following value for g:

$$g = \frac{\phi_a - \phi_\beta}{\log \dfrac{2c.\overline{BC}}{\beta.\overline{AC}}}.$$

Hence

$$\phi = \frac{\phi_a - \phi_\beta}{\log \dfrac{2c.\overline{BC}}{\beta.\overline{AC}}} \log \frac{r'}{r} + g'.$$

It is also possible to determine ϕ from relations (11) and (12), when the second electrode instead of being a closed cylindrical surface of infinitely small dimensions is likewise a circular cylindrical surface of finite radius. Here, as in the preceding problem, the only region that enters into consideration is that between the two electrodes. If the second electrode is completely enveloped by the first, as indicated in the annexed figure, and we

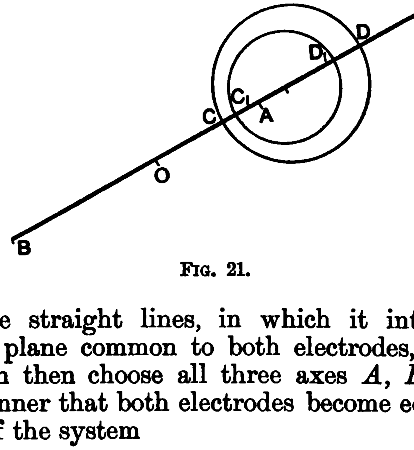

FIG. 21.

denote the straight lines, in which it intersects the diametral plane common to both electrodes, by C_1 and D_1, we can then choose all three axes A, B and O in such a manner that both electrodes become equipotential surfaces of the system

$$\phi = g \log \frac{r'}{r} + g',$$

where r and r' are the distances of any point from the axes A and B respectively.

In order that the above electrodes may be equipotential surfaces of the given system, it is evident that only the two following conditions need to be satisfied:

$$\overline{OC}\,.\,\overline{OD}=\overline{OA}^2 \text{ and } \overline{OC}_1\,.\,\overline{OD}_1=\overline{OA}^2,$$

or

$$\overline{OC}\,.\,\overline{OD}=\overline{OC}_1\,.\,\overline{OD}_1.$$

If we denote the unknown distance $\overline{OC}$ by ξ and express the remaining distances as function of this quantity and the known distances, we can write the last condition as follows:

$$\xi(\xi+\overline{CD})=(\xi+\overline{CC}_1)(\xi+\overline{CD}_1),$$

which gives the following value for ξ:

$$\xi=\frac{\overline{CC}_1\,.\,\overline{CD}_1}{\overline{CD}-\overline{CD}_1-\overline{CC}_1}=\frac{\overline{CC}_1\,.\,\overline{CD}_1}{\overline{DD}_1-\overline{CC}_1};$$

the positions of the axes A and B must then be given by the equation

$$\overline{OA}=\overline{OC}\,.\,\overline{OD}=\xi(\xi+\overline{CD}).$$

We see, therefore, that it is always possible to choose the axes A, B, and O in such a manner that all conditions are fulfilled.

The value of ϕ at any point of space, due to the presence of two eccentric cylindrical electrodes of finite radii, is therefore given by the function

$$\phi=g\log\frac{r'}{r}+g',$$

where r and r' denote the distances of the given point from the axes A and B respectively, and the latter lie in the diametral plane common to both electrodes at the distance

$$\frac{CC_1\,.\,\overline{CD}_1}{\overline{DD}_1-\overline{CC}_1}\left(\frac{\overline{CC}_1\,.\,\overline{CD}_1}{\overline{DD}_1-\overline{CC}_1}+\overline{CD}\right)$$

on either side of the axis O, which is situated on the same diametral plane at the distance

$$\frac{\overline{CC_1}\,.\,\overline{DD_1}}{\overline{DD_1}-\overline{CC_1}}$$

from the axis C (cf. figure 21).

For the determination of the constant g, we have the two equations

$$\phi_b = g\log\frac{\overline{BD}}{\overline{AD}}+g', \quad \text{and} \quad \phi_a = g\log\frac{\overline{BC_1}}{\overline{AC_1}}+g',$$

which give
$$g=\frac{\phi_a-\phi_b}{\log\dfrac{\overline{BC_1}\,.\,\overline{AD}}{\overline{AC_1}\,.\,\overline{BD}}}. \qquad\qquad (13)$$

Hence ϕ assumes the form

$$\phi=\frac{\phi_a-\phi_b}{\log\dfrac{\overline{BC_1}\,.\,\overline{AD}}{\overline{AC_1}\,.\,\overline{BD}}}\log\frac{r'}{r}+g'.$$

The current-strength at any point between the two electrodes will, of course, be the same here as in the preceding case, where the electrodes were closed cylindrical surfaces of infinitely small dimensions, provided the constant g has the same value in both cases. It is evident that the total current-flow through any cylindrical surface, enclosing either axis A or B, will be the same for all such (closed) surfaces; hence, to find the value of I for either electrode of the given system, we have only to determine its value for any such surface; that, for which the expression for I assumes the simplest form, is a circular cylinder, whose central axis coincides with the axis A or B. Formula (4) gives here

$$[I]=\int L\frac{d\phi}{dr}do=\frac{2\pi lL}{\log\dfrac{\overline{BC_1}\,.\,\overline{AD}}{\overline{AC_1}\,.\,\overline{BD}}}(\phi_a-\phi_b),$$

where l denotes the length of the cylindrical conductor.

The resistance W will therefore be

$$W = \frac{1}{2\pi l L} \log \frac{\overline{BC_1} \cdot \overline{AD}}{\overline{AC_1} \cdot \overline{BD}}.$$

Analogously, the total quantity of electricity R on either layer of a condenser, consisting of two circular but eccentric cylindrical shells, will be

$$[R] = \frac{lD}{2 \log \dfrac{\overline{BC_1} \cdot \overline{AD}}{\overline{AC_1} \cdot \overline{BD}}} (\phi_a - \phi_b). \quad \text{............(14)}$$

To confirm this value for R deduced from our analogy between electrostatics and stationary flow, we must evaluate the expression

$$[R] = \frac{1}{4\pi} \int D \frac{d\phi}{dn} do$$

for either electrode. The determination of $\frac{d\phi}{dn}$ at any point P of the electrode $r = ar'$ presents a few difficulties. We have

$$\frac{d\phi}{dn} = \frac{d\phi}{dx} \cos\beta + \frac{d\phi}{dy} \sin\beta, \quad \text{..............(15)}$$

where β is the angle between the normal n to the surface at the given point and the x-axis, as indicated in figure 22 on next page. To determine $\sin\beta$ and $\cos\beta$ as functions of x and y, we transform our equipotential surface (electrode) $r = ar'$ to rectilinear coordinates, and we find

$$x^2 + y^2 + c^2 - 2\mu c x = 0, \quad \text{.................(16)}$$

where $$\mu = \frac{1 + a^2}{1 - a^2}. \quad \text{.....................(17)}$$

Differentiating this equation, we have

$$(x - \mu c) dx + y dy = 0,$$

which gives
$$\frac{dy}{dx}=-\frac{x-\mu c}{y}=\text{tang}\,a, \quad\ldots\ldots\ldots\ldots(18)$$
where a is the angle between the tangent to the surface at the given point (P) and the x-axis (cf. figure 22). From the figure it follows, moreover, that
$$\text{tang}\,\beta=-\text{ctn}\,a;$$
we find therefore the following values for $\sin\beta$ and $\cos\beta$:
$$\sin\beta=\frac{y}{\sqrt{y^2+(x-\mu c)^2}},\quad \cos\beta=\frac{x-\mu c}{\sqrt{y^2+(x-\mu c)^2}}. \quad\ldots(19)$$

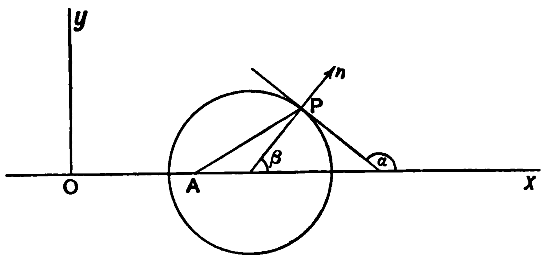

FIG. 22.

The values of $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ have already been determined on page 202. By the relation $r=ar'$ they can be written as follows:
$$\frac{d\phi}{dx}=-\frac{g(1-a^2)}{r^2}(x-\mu c),\quad \frac{d\phi}{dy}=-\frac{g(1-a^2)}{r^2}y. \quad\ldots(20)$$

By (19) and (20) the above expression (15) for $\frac{d\phi}{dn}$ can thus be written
$$\frac{d\phi}{dn}=-\frac{g(1-a^2)[(x-\mu c)^2+y^2]}{r^2\sqrt{y^2+(x-\mu c)^2}}.$$
Equation (16) gives
$$\sqrt{y^2+(x-\mu c)^2}=c\sqrt{\mu^2-1} \quad\ldots\ldots\ldots\ldots(21)$$

and $$r^2 = x^2 + y^2 + c^2 - 2cx = 2cx(\mu - 1),$$

by which the last expression for $\frac{d\phi}{dn}$ reduces to

$$\frac{d\phi}{dn} = -\frac{g(1-a^2)}{2x}\sqrt{\frac{\mu+1}{\mu-1}} = \frac{g(1-a^2)}{2ax};$$

our expression for R can thus be written

$$[R] = \frac{D}{4\pi}\int \frac{g(1-a^2)}{2ax}\, do,$$

where D has been assumed constant.

Replacing here the surface-element do by lds, where l is the length of the condenser, we have

$$[R] = \frac{D}{4\pi}\,\frac{g(1-a^2)l}{2a}\int \frac{ds}{x}. \quad \text{..................(22)}$$

To evaluate this integral, we write

$$ds = \sqrt{dx^2 + dy^2},$$

replace dy by its value from formula (18) and we have

$$ds = \frac{\sqrt{y^2 + (x - \mu c)^2}}{y}\, dx;$$

by equations (16) and (21) this expression can be written

$$ds = \frac{c\sqrt{\mu^2 - 1}}{\sqrt{-c^2 + 2\mu c x - x^2}}\, dx,$$

and the above integral thus assumes the following form:

$$\int \frac{ds}{x} = c\sqrt{\mu^2 - 1}\int \frac{dx}{x\sqrt{-c^2 + 2\mu c x - x^2}},$$

where the integration is to be extended round the circle

$r = ar$, that is, from $x = \frac{1+a}{1-a}c$ to $x = \frac{1-a}{1+a}c$ and back to $x = \frac{1+a}{1-a}c$. Explicitly, we thus have

$$\int \frac{ds}{x} = 2c\sqrt{\mu^2 - 1}\int_{\frac{1-a}{1+a}c}^{\frac{1+a}{1-a}c} \frac{dx}{x\sqrt{-c^2 + 2\mu cx - x^2}},$$

which integrated gives

$$\int \frac{ds}{x} = 2\sqrt{\mu^2 - 1}\left|\arcsin \frac{\mu x - c}{x\sqrt{\mu^2 - 1}}\right|_{\frac{1-a}{1+a}c}^{\frac{1+a}{1-a}c}.$$

This *arcsin* evidently assumes the value $-\frac{\pi}{2}$ at the lower limit and vanishes at the upper.

We thus find

$$\int \frac{ds}{x} = 2\pi\sqrt{\mu^2 - 1},$$

and hence by equation (22) the following value for R:

$$[R] = \frac{D}{2}\, g\frac{(1-a^2)l}{2a}\sqrt{\mu^2 - 1}.$$

Replacing here g and μ by their respective values (13) and (17), we finally get

$$[R] = \frac{lD}{2\log \frac{\overline{BC_1}\,.\,\overline{AD}}{\overline{AC_1}\,.\,\overline{BD}}}(\phi_a - \phi_b),$$

the expression (14) found above.

The capacity of our condenser is thus

$$\frac{lD}{2\log \frac{\overline{BC_1}\,.\,\overline{AD}}{\overline{AC_1}\,.\,\overline{BD}}}.$$

It is evident that the case, where two circular cylindrical electrodes lie entirely separated from each other, as indicated in the annexed figure, instead of the one completely enclosing the other, as above, can be treated exactly as the preceding problem, since the conditions that these electrodes be equipotential surfaces of the system

$$\phi = g \log \frac{r'}{r} + g',$$

namely that

$$\overline{OC} \,.\, \overline{OD} = \overline{OC_1} \,.\, \overline{OD_1} = c^2,$$

can always be fulfilled.

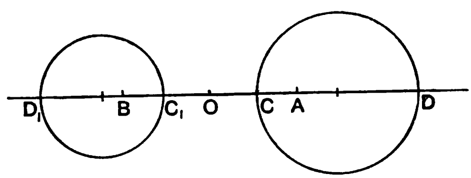

FIG. 23.

All the above solutions (integrals) for ϕ require that the value of I or R be the same, but opposite in sign for the two electrodes. We have already seen on pp. 190, 191, that this must be the case when the one electrode is enclosed by the other; it will not, however, necessarily follow, when the two electrodes lie entirely separated from each other, as in our last problem.

A second integral of the differential equation $\nabla^2\phi = 0$ is

$$\phi = g \log rr' + g', \qquad \text{(23)}$$

where r and r' are, as above, the bipolar coordinates of any point P.

To prove that this integral satisfies our differential equation we write ϕ as follows:

$$\phi = \tfrac{1}{2} g \log [(x-c)^2 + y^2] + \tfrac{1}{2} g \log [(x+c)^2 + y^2] + g'$$

and we have

$$\frac{d\phi}{dx}=g\left(\frac{x-c}{r^2}+\frac{x+c}{r'^2}\right),\quad \frac{d\phi}{dy}=gy\left(\frac{1}{r^2}+\frac{1}{r'^2}\right),\ \ldots\ldots(24)$$

$$\frac{d^2\phi}{dx^2}=g/r^2+g/r'^2-\frac{2(x-c)^2}{r^4}-\frac{2(x+c)^2}{r'^4}$$

$$\frac{d^2\phi}{dy^2}=g\left(\frac{1}{r^2}+\frac{1}{r'^2}\right)-2gy\left(\frac{1}{r^4}+\frac{1}{r'^4}\right),$$

which by relations (7) give

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0,$$

and hence $$\nabla^2\phi=0.$$

The equipotential surfaces of this new system (23) are evidently

$$rr'=a_1,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

surfaces of the 4th degree, the so-called lemniscates. As they are, however, of no special interest, instead of detaining the student here with their analytic analysis, we shall examine a property peculiar to their particular integral (23). We first determine the expression for $\frac{d\phi}{dn}$ at any point P on any equipotential surface $r=ar'$ of the first system. We have, as on p. 209,

$$\frac{d\phi}{dn}=\frac{d\phi}{dx}\cos\beta+\frac{d\phi}{dy}\sin\beta,$$

where $\sin\beta$ and $\cos\beta$ are given by formulae (19), but $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ by formulae (24); by the relation $r=ar'$ the latter can be written

$$\left.\begin{aligned}\frac{d\phi}{dx}&=\frac{g(1+a^2)}{r^2}\left(x-\frac{c}{\mu}\right),\\ \frac{d\phi}{dy}&=gy\frac{1+a^2}{r^2};\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(26)$$

and hence $\frac{d\phi}{dn}$ as follows:

$$\frac{d\phi}{dn}=\frac{g(1+a^2)\left[\left(x-\frac{c}{\mu}\right)(x-\mu c)+y^2\right]}{r^2\sqrt{y^2+(x-\mu c)^2}}.$$

By formula (16), which gives

$$\left(x-\frac{c}{\mu}\right)(x-\mu c)+y^2=x^2+y^2+c^2-cx\left(\mu+\frac{1}{\mu}\right)=cx\frac{\mu^2-1}{\mu},$$

and formula (21) this expression for $\frac{d\phi}{dn}$ can be written

$$\frac{d\phi}{dn}=\frac{g(1-a^2)}{2ac}, \quad \text{.....................(27)}$$

or, since $\frac{2ac}{1-a^2}$ is the radius of the circular cylindrical surface $r=ar'$, the given surface,

$$\frac{d\phi}{dn}=\frac{g}{\overline{CM}}$$

(cf. figure 20); it thus follows that $\frac{d\phi}{dn}$ has the same value at every point on any given surface $r=ar'$. Hence we have

$$[d\Pi]=L\frac{d\phi}{dn}do=\frac{gL}{\overline{CM}}do$$

(cf. formula (4)), for every surface-element do, that is, the same quantity of electricity passes through every surface-element do of any given equipotential surface $r=ar'$.

Similar investigations can be made for a third particular integral

$$\phi=g\log\frac{1}{rr'}+g' \quad \text{.....................(28)}$$

of our differential equation $\nabla^2\phi=0$; for, from the similarity of this expression to the one just considered, (23)—$g\log\frac{1}{rr'}=-g\log rr'$—it is evident, not only that

it will satisfy our differential equation, but that similar results can be deduced, namely, that $\frac{d\phi}{dn}$ will have the same value at every point on any given surface $r=ar'$, and hence that the same quantity of electricity will pass through every surface-element of that surface.

From the direction of the lines of flow it is evident that

$$\phi=g\log\frac{r'}{r}+g'$$

will be the solution of the given problem, when the electrode B is maintained at a higher and the electrode A at a lower potential than the intervening medium between them, that is, when electricity enters the system at the one electrode B and escapes from it at the other A.

The second integral

$$\phi=g\log rr'+g'$$

evidently corresponds to the case where electricity enters the system at both electrodes.

The third integral

$$\phi=g\log\frac{1}{rr'}+g'$$

is the solution of the given problem when electricity escapes from the system at both electrodes.

Let us next examine the following system. Any circular cylindrical surface is given, and electricity enters the system at any axis A within this surface. The position of the axis B, the so-called image of A, for which the given surface is an equipotential surface of the system

$$\phi=g\log\frac{r'}{r}+g',$$

is then given, as we have already seen on p. 203, by the relation

$$\overline{BM}=\frac{\overline{CM}^2}{\overline{AM}},$$

where the distances are referred to any straight line that passes through the central axis of the given surface and the given axis A cutting them at right angles. We have seen on the preceding page that if electricity enters the system at the axis B

$$\phi = g \log rr' + g',$$

and that the same quantity of electricity will then pass through every surface-element of the given surface.

The above will also hold for any other axis A_1, that lies within the given surface; its corresponding axis, or image B_1, will be given by the similar relation

$$\overline{B_1M} = \frac{\overline{CM}^2}{\overline{A_1M}}.$$

If electricity escapes from the system at both axes A_1 and B_1, the solution for ϕ will be

$$\phi = g_1 \log \frac{1}{r_1 r_1'} + g_1',$$

where r_1 and r_1' denote the distances of the point P from these axes A_1 and B_1, as indicated in the annexed figure.

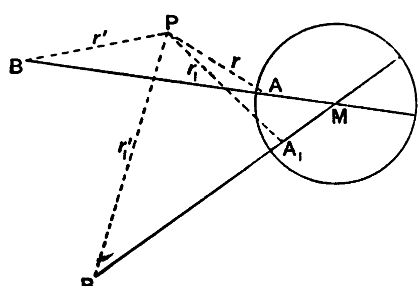

FIG. 24.

If now all four axes A, B, A_1, B_1 are present, and electricity enters the system at the former pair, and

escapes from it at the latter, it is evident that the value of ϕ at the point P must be given by the function

$$\phi = g \log rr' + g' - g_1 \log \frac{1}{r_1 r_1'} - g_1'.$$

If we next assume that the same quantity of electricity enters the system at the axes A and B as escapes from it at the axes A_1 and B_1, we have

$$[I] = [I_1],$$

hence

$$2\pi l L g = 2\pi l L g_1, \qquad (29)$$

which gives $g = g_1$. ϕ thus assumes the form

$$\phi = g \log \frac{rr'}{r_1 r_1'} + \text{const.} \qquad (30)$$

Moreover, since $g = g_1$, the same quantity of electricity would pass out through every surface-element do of the given surface due to the presence of the electrodes A and B, as would pass in through it due to that of the electrodes A_1 and B_1, that is, the resultant flow of electricity through any surface-element of the given surface will be zero; this follows directly from the differentiation of the value (30) for ϕ with regard to n, which gives $\frac{d\phi}{dn} = 0$ and hence $dI = 0$. We could, therefore, conceive the system divided by the given surface into two regions, and entirely neglect the presence of the outer region or that containing the electrodes B and B_1, without disturbing in any respect the flow of electricity within the inner region. Hereby we have thus found the solution for the flow of electricity through any circular cylinder due to the presence of any two axial electrodes. To avoid the appearance of infinity in our formulae, it is, however, necessary here, as on p. 204, to replace the axes A and A_1 by cylindrical surfaces generated by infinitely short radii vectores. Let the mean

radii vectores of these surfaces or electrodes be ρ_0 and ρ_1 respectively. The constant g can then be determined as in the preceding problems; at any point on the electrode A we have

$$\phi_0 = g \log \frac{\rho_0 \overline{AB}}{\overline{AA_1}\,\overline{AB_1}} + \text{const.},$$

where ϕ_0 denotes the mean value of the potential for that electrode. Similarly, we have for the electrode B

$$\phi_1 = g \log \frac{\overline{AA_1}\,\overline{A_1B}}{\rho_1 \overline{A_1B_1}} + \text{const.}$$

These relations give

$$g = \frac{\phi_1 - \phi_0}{\log \dfrac{\overline{AA_1}^2\,\overline{AB_1}\,\overline{A_1B}}{\rho_0\rho_1 \overline{A_1B_1}\,\overline{AB}}}.$$

and hence $$\phi = \frac{\phi_1 - \phi_0}{\log \dfrac{\overline{AA_1}^2\,\overline{AB_1}\,\overline{A_1B}}{\rho_0\rho_1 \overline{A_1B_1}\,\overline{AB}}} \log \frac{rr'}{r_1 r_1'} + \text{const.}$$

By equation (30) the current-strength will therefore be

$$2\pi l L g = 2\pi l L \frac{\phi_1 - \phi_0}{\log \dfrac{\overline{AA_1}^2\,\overline{AB_1}\,\overline{A_1B}}{\rho_0\rho_1 \overline{A_1B_1}\,\overline{AB}}},$$

and hence the resistance

$$W = \frac{1}{2\pi l L} \log \frac{\overline{AA_1}^2\,\overline{AB_1}\,\overline{A_1B}}{\rho_0\rho_1 \overline{A_1B_1}\,\overline{AB}}.*$$

We have restricted ourselves in the above investigations to the problem of stationary flow in order to avoid a complication of ideas. The same equations and considerations hold, of course, for the electrostatic problem. Here two thin wires pass through a dielectric of circular

* Cf. Kirchhoff's "Ueber den Durchgang eines electrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmige;" Poggendorf's *Annalen*, v. 64, 1845; *Gesammelte Abhandlungen*, p. 11.

cylindrical configuration parallel to its axis and are maintained at constant potentials.

Lastly, let us examine a spherical system. Denoting by r and r' the distances of any point P from the point A and its image B respectively, we find that the following function satisfies our differential equation $\nabla^2\phi=0$:

$$\phi=g/r-g/ar'+g',$$

where a, g, and g' are arbitrary constants. To prove this we form the following expressions, observing that $r^2=(x-c)^2+y^2+z^2$ and $r'^2=(x+c)^2+y^2+z^2$:

$$\frac{d\phi}{dx}=-g\left(\frac{x-c}{r^3}-\frac{x+c}{ar'^3}\right),\quad \frac{d\phi}{dy}=-gy\left(\frac{1}{r^3}-\frac{1}{ar'^3}\right),$$

$$\frac{d\phi}{dz}=-gz\left(\frac{1}{r^3}-\frac{1}{ar'^3}\right),$$

$$\frac{d^2\phi}{dx^2}=-g\left(\frac{1}{r^3}-\frac{3(x-c)^2}{r^5}\right)-g\left(\frac{1}{ar'^3}-\frac{3(x+c)^2}{ar'^5}\right),$$

$$\frac{d^2\phi}{dy^2}=-g\left(\frac{1}{r^3}-\frac{1}{ar'^3}\right)+3gy^2\left(\frac{1}{r^5}-\frac{1}{ar'^5}\right),$$

$$\frac{d^2\phi}{dz^2}=-g\left(\frac{1}{r^3}-\frac{1}{ar'^3}\right)+3gz^2\left(\frac{1}{r^5}-\frac{1}{ar'^5}\right),$$

and we find

$$\nabla^2\phi=-3g\left[\frac{1}{r^3}-\frac{1}{ar'^3}-\frac{(x-c)^2+y^2+z^2}{r^5}\right.$$
$$\left.+\frac{(x+c)^2+y^2+z^2}{ar'^5}\right]=0.$$

Here $r=ar'$ is the equation of a spherical surface. It is evident that ϕ will be constant, equal to g', at every point on this surface, which is quite a different result from that obtained for $\phi=\pm g\log rr'+g'$ on the surface of the circular cylinder $r=ar'$ of the preceding problem. If now any given quantity of electricity g is placed at

the point A, and the quantity $-g/a$ at its image B, ϕ will thus be constant and equal to g'—g' is the value, which ϕ will assume at infinite distance, at every point on the surface $r = ar'$, that is, the spherical surface will be an equipotential surface of the given system By equation (11) we have

$$\frac{\overline{AC}}{\overline{BC}} = \frac{c - \overline{OC}}{c + \overline{OC}} = \frac{c\left(1 - \frac{1-a}{1+a}\right)}{c\left(1 + \frac{1-a}{1+a}\right)} = a.$$

Consequently, when the spherical surface, that is, its points of intersection C and D with the x-axis, the point A, and hence its image B are given, the quantity of electricity in B will be determined by the relation

$$g/a = g\frac{\overline{BC}}{\overline{AC}}.$$

This theorem gives us directly the electrification of a given sphere in electric communication with the earth due to the presence of any quantity of electricity concentrated at any point of space.*

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* Cf. Thomson's "Paper on Electrostatics," *Liouville Journal*, 1845, 1847, and Maxwell's treatise chapter XI., etc.

CHAPTER XI.

SECTION XXV. MAGNETIC PHENOMENA WHERE ELECTRIC PHENOMENA EITHER DO NOT APPEAR OR ARE ELECTROSTATIC.

In the preceding chapters we have eliminated α, β, γ from our equations. In the following we shall retain these quantities and investigate the phenomena to which they give rise.

Differentiate the fundamental equations (10, II.), the first with regard to x, the second to y, and the third to z, add, and we get

$$\frac{d}{dt}\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]=0\,; \quad \text{..........(1)}$$

this equation is general.

As we are restricting ourselves to electrostatic phenomena, all quantities will be independent of the time; since P, Q, R vanish in conductors, and $L=0$ for insulators, our fundamental equations (9, II.) reduce here to

$$\frac{d\beta}{dz}=\frac{d\gamma}{dy},\quad \frac{d\gamma}{dx}=\frac{d\alpha}{dz},\quad \frac{d\alpha}{dy}=\frac{d\beta}{dx},$$

from which it follows that α, β, γ may be regarded as the partial derivatives with regard to the coordinates of a function ψ, known as the *magnetic potential*; we put

$$\alpha=-\frac{d\psi}{dx},\quad \beta=-\frac{d\psi}{dy},\quad \gamma=-\frac{d\psi}{dz}. \quad \text{.........(2)}$$

By these formulae (2) the expression (1) can be written

$$\frac{d}{dt}\left[\frac{d}{dx}\left(M\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(M\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(M\frac{d\psi}{dz}\right)\right]=0, \quad \text{.....(3)}$$

hence $$\frac{d}{dx}\left(M\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(M\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(M\frac{d\psi}{dz}\right)=\text{const.}$$

We shall call the quantity

$$-\frac{1}{4\pi}\left[\frac{d}{dx}\left(M\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(M\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(M\frac{d\psi}{dz}\right)\right]$$

the *density of the real magnetism* and denote it by η_r.

By the principle of the continuity of transitions, this expression will evidently assume the following form on the dividing-surface of adjoining media:

$$M_0\frac{d\psi_0}{dn}-M_1\frac{d\psi_1}{dn}=\text{H}_r, \quad \text{..................(4)}$$

(cf. § 5). It follows now from our theorem from the theory of the potential (cf. § 21) that ψ is determined uniquely as a function of the coordinates by these two equations (3) and (4), provided ψ and its derivatives are only finite and continuous throughout the given region; these conditions can be taken for granted for reasons similar to those already stated on pp. 105 and 167.

If ψ is a function of x, y, z, the density of the real magnetism η_r will in general be different from zero at any given point of space. Since this density cannot vary with the time (cf. above), the real magnetism must have existed at the given point since eternity and will thus remain there for ever; our only other alternative would be to assume that our fundamental equations are not always valid. In accepting the latter, we should be able to account for the creation of real magnetism in certain bodies, as iron; whenever real magnetism were once so created, it would necessarily continue to exist until our fundamental equations again ceased to be valid. This is now precisely the manner in which real electricity behaves in dielectrics, where, namely, $L=X=Y=Z=0$

(cf. §§ 6 and 14); as the equations themselves are similar in both cases, any desired expression of the one can always be obtained from the corresponding expression of the other by means of the substitution

$$\begin{pmatrix} M, & \alpha, & \beta, & \gamma, & \psi \\ D, & P, & Q, & R, & \phi \end{pmatrix}.$$

We should, however, observe the following essential difference in the behaviour of the densities ϵ_r and η_r. We have seen in § 6 that the density ϵ_r remains constant at every point, where L, X, Y, and Z vanish; on the other hand, whenever the external electromotive forces* do not vanish, the products LX, LY, LZ* must be retained in our fundamental equations (9, II.) and these will evidently give rise to a variation in the density ϵ_r, that is, real electricity will be either created or destroyed wherever external electromotive forces appear. Such a variation in the density η_r is now quite impossible, since external magnetomotive forces are entirely unknown to Maxwell's equations (cf. equations (10, II.)). Observe that in the investigations of § 6, concerning the behaviour of ϵ_r in insulators, we have assumed that $L=X=Y=Z=0$.

On account of the above similarity between ϵ_r and η_r, we can likewise illustrate magnetic phenomena by conceiving that all bodies contain a positive and a negative (magnetic) fluid, which behave like the electric fluids of dielectrics. There is, however, nothing in magnetism that corresponds to the behaviour of the electric fluids in conductors, that is, to their flow. In order that our concrete representation may agree in every respect with the current theory, according to which the positive and negative fluids of permanent magnets can only be moved with the greatest difficulty, we should have to suppose that equal quantities of real positive and negative magnetism had entered the magnets at some past period, when our fundamental equations were invalid, but that

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* Cf. foot-note, p. 22, and text, p. 39.

their neutral magnetisms always obey the laws that hold for soft iron. The current theory could also be explained by assuming that M is approximately equal to unity in steel magnets, and that the air is non-magnetizable, whereby the neutral magnetism would be rendered almost immovable (cf. formula (9) and following text). In this case enormous external magnetomotive forces —our fundamental equations would not remain valid during their action—would be required to produce a powerful permanent magnet.

As P, Q, R were defined as the components of the electric force that acts on unit quantity of real electricity, let α, β, γ be those of the magnetic force that acts on unit quantity of real magnetism. Just as the apparent action at a distance between two given quantities of real electricity was, as we have seen in § 16 (cf. formula (12)), inversely proportional to D, the constant of electric conduction, so it is easy to show that the action between two given quantities of real magnetism is inversely proportional to M. As D and M are analogous quantities we shall call M the constant of magnetic conduction (cf. also p. 192).

In electrostatics the electromotive forces P, Q, R were, of course, entirely independent of the value of the arbitrary constant $\mathfrak{e}$, which was introduced, among other reasons, to effect an agreement in our concrete representation between the phenomena of electrostatics and the Hertzian oscillations (cf. pp. 135-136). This was not, however, true of the electric polarization or induction (cf. § 14); for example, the electric polarization vanished for the standard medium $D=1$, when $\mathfrak{e}$ was assigned the value unity (cf. p. 132). As our above equations for magnetism are analogous to those for the behaviour of electricity in a dielectric, we can likewise introduce here a similar arbitrary constant $\mathfrak{m}$, and define the following quantity as the *density of the free magnetism*:

$$\eta' = \frac{\mathfrak{m}}{4\pi}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right); \qquad (5)$$

similarly—compare our general definition for ϵ_f on p. 108, this definition shall be retained when equations (2) do not hold. Observe, on the other hand, that the density of the real magnetism

$$\eta_r = \frac{1}{4\pi}\left[\frac{d}{dx}(M\alpha)+\frac{d}{dy}(M\beta)+\frac{d}{dz}(M\gamma)\right]. \qquad (6)$$

is independent of the constant $\mathfrak{m}$.

For the case under consideration, where, namely, the ether has subsided to the electrostatic state and equations (2) thus hold, we have

$$\eta_f = -\frac{\mathfrak{m}}{4\pi}\nabla^2\psi, \qquad (7)$$

and

$$\eta_r = -\frac{1}{4\pi}\left[\frac{d}{dx}\left(M\frac{d\psi}{dx}\right)+\frac{d}{dy}\left(M\frac{d\psi}{dy}\right)+\frac{d}{dz}\left(M\frac{d\psi}{dz}\right)\right]. \qquad (8)$$

If M is assumed to be constant, these formulae give the following relation between η_f and η_r:

$$\eta_f = \frac{\mathfrak{m}}{M}\eta_r. \qquad (9)$$

Similarly—compare § 14, the components α, β, γ of the magnetic force, but not those of the magnetic polarization, will be independent of the arbitrary constant $\mathfrak{m}$.

The magnetic moment per unit-volume along the x-axis is evidently

$$\frac{M-\mathfrak{m}}{4\pi}\alpha. \qquad (10)$$

(cf. formula (16, VI.)). If we assign $\mathfrak{m}$ the value zero, the total magnetic energy V will alone be due to magnetic polarization, since by formula (5) the density of the free magnetism η_f vanishes at every point. The magnetic moment per unit-volume along the x-axis then becomes

$$\frac{M}{4\pi}\alpha;$$

Maxwell calls the quantity $M\alpha$ the magnetic induction and writes

$$M\alpha = a$$

(cf. p. 193).

For air, $M = 1$, $\qquad a_a = \alpha.$

If we assume that the air is insusceptible to magnetic polarization or induction and wish to characterize other bodies by their magnetic inductibility referred to air, we put $\mathfrak{m} = 1$; Maxwell calls this magnetic moment the strength of the magnetization and writes

$$A = \frac{M-1}{4\pi}\alpha,$$

which for air, $M = 1$, vanishes; von Helmholtz calls the quantity A the magnetic moment (per unit-volume) and denotes it by λ; he writes

$$\lambda = \theta\alpha, \quad \text{where} \quad \theta = \frac{M-1}{4\pi},$$

hence

$$M = 1 + 4\pi\theta. \qquad \text{(11)}$$

(cf. § 37).

The above relations give

$$\alpha + 4\pi A = a. \qquad \text{(12)}$$

This equation can be interpreted physically. Take any body under the action of magnetic forces, and determine the magnetic force exerted by it on a magnetic pole of unit strength placed at any point of the body. If the body is a fluid, the magnetic pole can be immersed in it, and it is evident that the force acting on the pole in the direction of the x-axis will be α. This corresponds exactly to von Helmholtz's assumption that, when a body is charged with real electricity and immersed in a fluid, the force acting on it in the direction of the x-axis is equal to the component P of the electromotive force along that axis times its electric charge; he supposes that the electrically polarized fluid forms a kind of

stationary medium of its own, which does not follow the immersed body in its course, but yields to the slightest displacement of the latter within it, and that the fluid thus exercises no force whatever on the immersed body (cf. § 17, pp. 136-138). As the immersion of a magnetic pole in a solid is impossible, our only alternative here would be to bore a small hole in the body to the given point, and place the magnetic pole at that point. If the magnetic pole were already contained in the body, we could not even then dispense with the hole, since it would otherwise be impossible to observe the action of the magnetic forces upon it. We shall see directly that the force that the magnetic field exerts on the magnetic pole depends upon the form of the hole, which is a most remarkable result, since this is not true of gravitating masses. The component along the x-axis of the magnetic force that acts on magnetic pole of unit strength is in fact only a, when the hole has the form of a very long thin cylinder, whose axis runs parallel to the x-axis, since it is in this case only that the action along the x-axis of the magnetism residing within the walls of the cylindrical hole is infinitesimal compared to that of the magnetic field itself. The appearance of magnetism within the walls of any hole is, of course, due to the removal of an equal quantity of magnetism of the opposite kind in boring the hole—compare the analogous conception of the electric polarization on p. 109; the material removed from the hole is usually replaced by air, $M=1$, which becomes non-magnetizable, when the arbitrary constant $\mathfrak{m}$ is assigned the value unity. It is evident that the only magnetism within the walls of the cylindrical hole that exerts any component-force along the x-axis on the given magnetic pole is that within its two ends, and that, as this force is infinitesimal in comparison to that exerted by the field, it may be neglected.

We find quite a different result, when the axis of the cylindrical hole is taken at right angles to the x-axis

instead of parallel to it as above. Here it is the real magnetism within the sides of the cylinder and not at its distant ends, that exerts a component force along the x-axis, and this action cannot be neglected as above. Let the cross-section of the cylinder be rectangular, and its shorter side be parallel to the x-axis. If A denotes the density of the real magnetism within the one wall of the hole parallel to the $x=0$ plane and q its area, Aq will be the quantity of real magnetism within that wall; an equal quantity of real magnetism of the opposite kind will reside within the opposite wall. If δ denotes the distance between these two walls, $Aq\delta$ will be the magnetic moment along the x-axis of the real magnetism within them. Since the real magnetism within the walls of the cylinder parallel to the $y=0$ and $z=0$ planes has no moment along the x-axis, $Aq\delta$ will be the magnetic moment along that axis of all the real magnetism within the walls of the hole, that is, within its whole inner surface. The total magnetic moment per unit-volume along the x-axis will therefore be

$$\frac{Aq\delta}{q\delta}=A.$$

To find the component along the x-axis of the magnetic force exerted by the real magnetism within the hole on the magnetic pole P, we drop a perpendicular p from P on either wall of the cylinder, as indicated in the annexed figure. The quantity of real magnetism within any element $2\pi r dr$ of this wall formed by describing about the point O, where the perpendicular p meets the wall, two circles of radii r and $r+dr$ is evidently

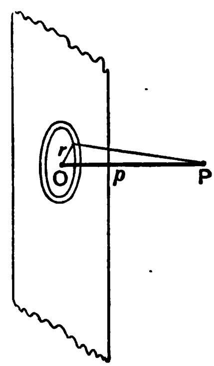

FIG. 25.

$$2\pi r dr \,.\, A.$$

The total force exerted by this real magnetism on our

magnetic pole P is thus

$$\frac{2\pi r dr A}{r^2+p^2}$$

and hence its component along the x-axis

$$\frac{2\pi r dr A p}{(r^2+p^2)^{\frac{3}{2}}}.$$

The total component force along the x-axis due to the presence of all the real magnetism in either wall is therefore

$$\int_0^\infty \frac{2\pi r dr A p}{(p^2+r^2)^{\frac{3}{2}}} = 2\pi A p \int_0^\infty d\left(\frac{1}{\sqrt{p^2+r^2}}\right) = 2\pi A.$$

In choosing ∞ as the upper limit of this integral, we assume that on the boundaries of the given surface or wall r is infinitely large in comparison to the distance p of the magnetic pole P from the former.

It follows from the value of the above integral that the component of the total magnetic force along the x-axis is entirely independent of the distance p of the magnetic pole from either wall. The density of the real magnetism in the opposite wall of the cylinder is $-A$; similarly, the component along the x-axis of the force exerted on the pole P by its real magnetism is

$$2\pi A.$$

We have seen that the real magnetism within the other two walls and the two ends of the cylinder exerts no component force along the x-axis. The component along the x-axis of the force exerted by the magnetic field itself on unit pole is a; the component of the total magnetic force along this axis will thus be

$$a + 4\pi A = \alpha.$$

By Poisson's theory of magnetism the magnetic force a can be found for any form of hole; it has indeed been

actually determined for several (given) holes by Maxwell and Kelvin.

Observe here that the corresponding equation in the theory of electric induction, namely,

$$P + 4\pi\epsilon_{vH} = \Pi,$$

where ϵ_{vH} denotes the electric moment per unit-volume, can be similarly interpreted (cf. end of § 17).

Maxwell assumes that his medium-constant μ, which is our $(M - \mathfrak{m})$, is $\gtrless 0$ for air, that is, that when the air is under the action of magnetic forces it contains a certain amount of magnetic energy V, and, moreover, that it is this energy that brings about the action of electro-dynamic and magnetic forces at a distance. We can now either follow Maxwell and assume that the air is susceptible to magnetic polarization, or we can put $\mathfrak{m} = \dot{D} = 1$ (in our concrete representation) and conceive the air as non-polarizable, as in the theory of action at a distance. In the latter case all other bodies would have to be conceived as susceptible to magnetic polarization, and A would thus become the magnetic moment (per unit-volume) of the given body as in the old theory, where the air is likewise assumed to be non-polarizable and the magnetic forces are conceived as acting directly at a distance; we should then have to define A as the magnetic moment (per unit-volume) of the given body referred to air, for which $A_a = 0$. For a long thin cylinder, whose axis is parallel to the x-axis, we should thus have

$$A_l = \theta a_a = \frac{M-1}{4\pi} a_a, \qquad \text{(13)}$$

where a_a denotes the magnetic force of the field before the introduction of the cylinder. For a short cylinder, similarly placed, the component of the magnetic force along the x-axis would not be a_a, but

$$a_a - 4\pi A_s$$

—compare the annexed figure, where A_s denotes the moment of the given cylinder; A_s is also the density of the real magnetism within its end. The determination

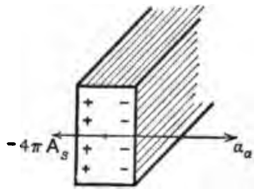

FIG. 26.

of these two component forces is precisely similar to that of the magnetic force a of formula (12).

The magnetic moment of the short cylinder is thus

$$A_s = \theta(a_a - 4\pi A_s) = \frac{\theta a_a}{M},$$

or, by formula (13), $$A_s = \frac{A_l}{M}.$$

The constant M can, therefore, be defined as the quotient A_l/A_s. It is evident that the value of the magnetic moment A for any other form of the given body—for example, a sphere—will lie between the values A_l and A_s; its actual determination can be found by Poisson's theory of magnetism.

In the following we shall put $\mathfrak{m} = 1$ and write

$$\eta_f = \frac{1}{4\pi}\left(\frac{da}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = -\frac{1}{4\pi}\nabla^2\psi, \quad \text{.........(14)}$$

from which it follows that

$$\psi = \int \frac{\eta_f d\tau}{\rho} \quad \text{.......................(15)}$$

(cf. formula (11, VI.)).

As in dielectrics we imagine lines of electric induction (cf. p. 192), let us conceive here a system of curves or lines

of magnetic force, or, better, of magnetic induction, that coincide at every point with the direction of the vector $(M\alpha, M\beta, M\gamma)$, the density of these curves, that is, the number of curves passing through unit surface at right angles to their direction, being always chosen equal to the value of this vector. The excess of the number of lines of magnetic induction that leave any volume-element $dx\,dy\,dz$ over that of those that enter it, that is, the number of lines created within it, will evidently be

$$\left[\frac{d}{dx}(M\alpha)+\frac{d}{dy}(M\beta)+\frac{d}{dz}(M\gamma)\right]dx\,dy\,dz$$

(cf. the analogous expression on p. 193), or, by formula (6),

$$4\pi\eta_r\,dx\,dy\,dz\,;$$

it thus follows that, if η_r vanishes, lines of magnetic induction can neither end nor begin in the given volume-element.

The force acting between two quantities of real magnetism m_r and m_r' in a medium, where M is constant, will evidently be

$$m_r a = m_r' a' = \frac{m_r m_r'}{M\rho^2} \quad \text{.................} (16)$$

(cf. the analogous expression (12, VII.)). If $\psi_p, \alpha_p, \beta_p, \gamma_p$, denote the values of ψ, α, β, γ respectively at the distance ρ from the magnetic pole m_r, it follows then that

$$\psi_p = \frac{m_r}{M\rho}, \quad \sqrt{\alpha_p^2+\beta_p^2+\gamma_p^2} = -\frac{d\psi_p}{d\rho} = \frac{m_r}{M\rho^2}.$$

The number of lines of magnetic induction that start from the magnetic pole m_r will evidently be

$$Z = 4\pi\int\eta_r d\tau = 4\pi m_r \quad \text{.................} (17)$$

The assumption that our fundamental equations are not valid at certain periods—this is the only way in

which real magnetism could be created—appears at first sight quite feasible. We might perhaps suppose that our above equations hold only for bodies at rest and that Maxwell's equations for moving bodies render the creation of real magnetism possible; we shall see, however, later that this is not the case (cf. § 42). We should not, moreover, fail to observe here that quite essential corrections must be applied to our fundamental equations in order that they may hold for many magnetizable bodies, and, indeed, for the most important ones, as iron and steel—for these highly magnetizable bodies our equations cease, namely, to be linear; Maxwell does, in fact, introduce such corrections or terms in order to account for the creation of real magnetism in these bodies. It is, however, a matter of no little moment to found such an important conception as that of magnetism on the assumption that Maxwell's equations cease to remain valid at certain periods or perhaps indeed never hold. The importance of such a step becomes even more apparent when we recall our mechanical conception of electricity and magnetism in Chapter I., in accordance with which and formulae (10, I.), which were taken as our original definitions of the quantities α, β, γ, real magnetism must be excluded from all bodies or media.

The above difficulty can be surmounted by introducing Ampère's hypothesis of molecular currents into Maxwell's theory; according to this hypothesis magnetism is only possible when electric currents are present and the magnetic properties of steel magnets are attributed to the presence of molecular currents within them.

In order that the following investigations may be as general as possible, we shall, however, assume that only equations (10, II.), and not the more special equations or definitions (10, I.), are valid, and thus not exclude the possibility of real magnetism residing at given points—the real magnetism will of course be entirely independent of the time; later, if desired, it can always be put equal to zero.

SECTION XXVI. MAGNETIC PHENOMENA FOR THE STATE OF STATIONARY FLOW, REAL MAGNETISM NOT EXCLUDED.

We have by formulae (13, III.)

$$p=L(P+X),\quad q=L(Q+Y),\quad r=L(R+Z), \ldots\ldots(18)$$

where in the present case, that stationary flow, P, Q, R and X, Y, Z are functions of the coordinates only, being entirely independent of the time. Equations (9, II.) can, therefore, be written

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=\frac{4\pi p}{\mathfrak{A}},\quad \frac{d\gamma}{dx}-\frac{d\alpha}{dz}=\frac{4\pi q}{\mathfrak{A}},\quad \frac{d\alpha}{dy}-\frac{d\beta}{dx}=\frac{4\pi r}{\mathfrak{A}}.$$

Differentiate the second of these equations with regard to z and the third to y, subtract, and we get

$$\nabla^2\alpha-\frac{d}{dx}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=\frac{4\pi}{\mathfrak{A}}\left(\frac{dr}{dy}-\frac{dq}{dz}\right)$$

and similarly

$$\left.\begin{aligned}\nabla^2\beta-\frac{d}{dy}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)&=\frac{4\pi}{\mathfrak{A}}\left(\frac{dp}{dz}-\frac{dr}{dx}\right)\\ \nabla^2\gamma-\frac{d}{dz}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)&=\frac{4\pi}{\mathfrak{A}}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)\end{aligned}\right\}\ldots\ldots(19)$$

Here p, q, r are to be regarded as known quantities and the values of α, β, γ are sought.

From these equations (19) it follows that

$$\alpha=\alpha_1+\alpha_2,\quad \beta=\beta_1+\beta_2,\quad \gamma=\gamma_1+\gamma_2, \ldots\ldots\ldots(20)$$

where

$$\left.\begin{aligned}\alpha_1=\frac{1}{\mathfrak{A}}\int\left(\frac{dq}{dz'}-\frac{dr}{dy'}\right)\frac{d\tau'}{\rho},\quad \beta_1=\frac{1}{\mathfrak{A}}\int\left(\frac{dr}{dx'}-\frac{dp}{dz'}\right)\frac{d\tau'}{\rho},\\ \gamma_1=\frac{1}{\mathfrak{A}}\int\left(\frac{dp}{dy'}-\frac{dq}{dx'}\right)\frac{d\tau'}{\rho},\end{aligned}\right\}\ldots(21)$$

$$\left.\begin{aligned}\alpha_2 &= -\frac{1}{4\pi}\int\frac{d}{dx'}\left(\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}\right)\frac{d\tau'}{\rho},\\ \beta_2 &= -\frac{1}{4\pi}\int\frac{d}{dy'}\left(\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}\right)\frac{d\tau'}{\rho},\\ \gamma_2 &= -\frac{1}{4\pi}\int\frac{d}{dz'}\left(\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}\right)\frac{d\tau'}{\rho},\end{aligned}\right\}\quad\text{.........(22)}$$

and $$\rho^2=(x-x')^2+(y-y')^2+(z-z')^2,$$

where (x', y', z') denotes any point of space and (x, y, z) that at which the values of α, β, γ are sought.

The integral

$$\int\frac{dq}{dz'}\frac{d\tau'}{\rho}=\iiint\frac{dq\,dx'dy'}{\rho}$$

integrated by parts gives

$$\iint dx'dy'\left|\frac{q}{\rho}\right|-\iiint q\,dx'dy'd\left(\frac{1}{\rho}\right).$$

The first of these integrals vanishes, since, by the principle of the continuity of transitions, the confines of space, $\rho=\infty$, can always be taken as the limits of integration. The second integral can be written as follows by the relation

$$\frac{d\left(\frac{1}{\rho}\right)}{dz'}=-\frac{d\left(\frac{1}{\rho}\right)}{dz}:\quad\text{.....................(23)}$$

$$\iiint q\,dx'dy'dz'\frac{d\left(\frac{1}{\rho}\right)}{dz}=\frac{d}{dz}\int\frac{q\,d\tau'}{\rho}.\quad\text{...........(24)}$$

The other integrals of formulae (21) and (22) can be similarly transformed, and the formulae themselves thus written as follows:

$$\left.\begin{aligned} \alpha_1 &= \frac{1}{\mathfrak{B}}\left[\frac{d}{dz}\int\frac{q\,d\tau'}{\rho} - \frac{d}{dy}\int\frac{r\,d\tau'}{\rho}\right] = \frac{1}{\mathfrak{B}}\left[\frac{d\bar{q}}{dz} - \frac{d\bar{r}}{dy}\right], \\ \beta_1 &= \frac{1}{\mathfrak{B}}\left[\frac{d\bar{r}}{dx} - \frac{d\bar{p}}{dz}\right], \quad \gamma_1 = \frac{1}{\mathfrak{B}}\left[\frac{d\bar{p}}{dy} - \frac{d\bar{q}}{dx}\right] \end{aligned}\right\} \text{......(25)}$$

where

$$\bar{p} = \int\frac{p\,d\tau'}{\rho}, \quad \bar{q} = \int\frac{q\,d\tau'}{\rho}, \quad \bar{r} = \int\frac{r\,d\tau'}{\rho}, \text{.........(25}a\text{)}$$

and

$$\left.\begin{aligned} \alpha_2 &= -\frac{1}{4\pi}\frac{d}{dx}\int\left(\frac{d\alpha}{dx'} + \frac{d\beta}{dy'} + \frac{d\gamma}{dz'}\right)\frac{d\tau'}{\rho} = -\frac{d}{dx}\int\frac{\eta_f\,d\tau'}{\rho} \\ &= -\frac{d\psi}{dx} \\ \beta_2 &= -\frac{d\psi}{dy}, \quad \gamma_2 = -\frac{d\psi}{dz}, \end{aligned}\right\} \text{..(26)}$$

where

$$\psi = \int\frac{\eta_f\,d\tau'}{\rho}. \text{........................(27)}$$

ψ is here the so-called magnetic potential; although it is analytically identical with the electrostatic magnetic potential, we should observe the following characteristic difference between them: we have seen in the preceding article that for electrostatic phenomena the magnetic forces α, β, γ were the partial derivatives with regard to the coordinates of a potential function ψ defined by formula (15), just as for the aphotic (electrostatic) state P, Q, R were the partial derivatives with regard to the coordinates of a potential function ϕ defined by formula (11, VI.), whereas in the present case, that of stationary flow, the magnetic forces α, β, γ are functions not only of the magnetic potential ψ, which is defined by the same formula (27) as in the electrostatic case, but of the so-called vector potentials $\bar{p}$, $\bar{q}$, $\bar{r}$, defined by formulae (25a); these vector potentials are the components of the vector $\sqrt{\bar{p}^2+\bar{q}^2+\bar{r}^2}$. The densities p, q, r are known

as the components of the so-called vector density $\sqrt{p^2+q^2+r^2}$.

Replace next $\frac{d\left(\frac{1}{\rho}\right)}{dz}$ by its value $-\frac{z-z'}{\rho^3}$ in equation (24), and we have

$$\int q\frac{d\left(\frac{1}{\rho}\right)}{dz}d\tau' = -\int\frac{q(z-z')}{\rho^3}d\tau' = \frac{d}{dz}\int\frac{q\,d\tau'}{\rho},$$

by which and similar transformations formulae (25) and (26) can be written in the following useful form:

$$a_1 = \frac{1}{4\pi}\int\frac{d\tau'}{\rho^3}[r(y-y')-q(z-z')], \qquad\text{.............(28)}$$

$$a_2 = \frac{1}{4\pi}\int\frac{d\tau'}{\rho^3}\left(\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}\right)(x-x'), \qquad\text{.........(29)}$$

and similar expressions for β_1, β_2, and γ_1, γ_2.

Lastly, the expression

$$\frac{d\bar{p}}{dx}+\frac{d\bar{q}}{dy}+\frac{d\bar{r}}{dz}$$

or, explicitly

$$\frac{d\bar{p}}{dx}+\frac{d\bar{q}}{dy}+\frac{d\bar{r}}{dz} = \frac{d}{dx}\int\frac{p\,d\tau'}{\rho}+\frac{d}{dy}\int\frac{q\,d\tau'}{\rho}+\frac{d}{dz}\int\frac{r\,d\tau'}{\rho},$$

can be written as follows by relation (23) and its two analogous relations:

$$\frac{d\bar{p}}{dx}+\frac{d\bar{q}}{dy}+\frac{d\bar{r}}{dz} = -\int\left[p\frac{d\left(\frac{1}{\rho}\right)}{dx'}+q\frac{d\left(\frac{1}{\rho}\right)}{dy'}+r\frac{d\left(\frac{1}{\rho}\right)}{dz'}\right]d\tau',$$

or, integrated by parts:

$$\frac{d\bar{p}}{dx}+\frac{d\bar{q}}{dy}+\frac{d\bar{r}}{dz} = \int\frac{d\tau'}{\rho}\left(\frac{dp}{dx'}+\frac{dq}{dy'}+\frac{dr}{dz'}\right).$$

Since the state of stationary flow is now characterized among other conditions by the following (cf. formulae (2, IX.)):

$$\frac{d}{dx'}L(P+X)+\frac{d}{dy'}L(Q+Y)+\frac{d}{dz'}L(R+Z)=0,$$

which, by formula (18), can be written

$$\frac{dp}{dx'}+\frac{dq}{dy'}+\frac{dr}{dz'}=0,$$

we thus have

$$\frac{d\bar{p}}{dx}+\frac{d\bar{q}}{dy}+\frac{d\bar{r}}{dz}=0. \quad\text{..................(30)}$$

If we do not assume the validity of equations (10, I.), that is, if we accept the possibility of the existence of real magnetism, α, β, γ will be the component-forces that act on unit quantity of real magnetism, as has been shown in the preceding article, where real magnetism alone is active. As these forces can depend only on the state of the ether in the immediate neighbourhood of the given point, that is, on the value of the potential ψ in its immediate proximity, and not upon the manner in which the electro-magnetic state has been produced, it will, of course, be immaterial whether the given state were brought about by the action of magnetic forces (α_1, β_1, γ_1) due to the presence of electric currents in the field, provided none pass through the given point, or whether it is to be attributed to the presence of real magnetism (α_2, β_2, γ_2); unless the above condition were fulfilled, the immediate neighbourhood of the magnetic pole might be differently constituted from the other regions, for $\left(\frac{1}{\rho}\right)$ and hence $\bar{p}$, $\bar{q}$, $\bar{r}$ become infinitely large in the former, and the validity of the above could thus not be concluded, until this region were more carefully examined.

If we exclude the possibility of real magnetism, equations (14), (20)–(22) and (25)–(27) still remain valid,

provided, of course, we write $\eta_r=0$, but the physical meaning of the quantities α, β, γ must then be sought. Since

$$\eta_r=\frac{1}{4\pi}\left(\alpha\frac{dM}{dx}+\beta\frac{dM}{dy}+\gamma\frac{dM}{dz}\right)+M\eta_f,$$

we could then write

$$\eta_f=-\frac{1}{4\pi M}\left(\alpha\frac{dM}{dx}+\beta\frac{dM}{dy}+\gamma\frac{dM}{dz}\right);$$

η_f is here the density of the free magnetism generated by the action of magnetic forces due to electric currents either in regions where M is variable, or on dividing-surfaces between bodies, for which M has different values; this magnetism does not, of course, give rise to either real or free magnetism proper.

Let us apply the above general investigations to a system of linear conductors or wires traversed by electric currents. Let ds' be any linear element of the given conductor, dx', dy', dz' its projections on the coordinate axes, i' the current-strength at the point (x', y', z') measured in electrostatic units, and ρ the distance between that point and the point (x, y, z), at which the values of α, β, γ are sought; let the direction-cosines of the vector ρ be denoted by l, m, n and those of the tangent to the conductor at the point (x', y', z') by λ, μ, ν. We have then

$$l=\frac{x-x'}{\rho},\quad m=\frac{y-y'}{\rho},\quad n=\frac{z-z'}{\rho},$$

$$\lambda=\frac{dx'}{ds'},\qquad \mu=\frac{dy'}{ds'},\qquad \nu=\frac{dz'}{ds'},$$

and

$$p=\frac{\lambda i'}{\sigma},\quad q=\frac{\mu i'}{\sigma},\quad r=\frac{\nu i'}{\sigma},\quad \sqrt{p^2+q^2+r^2}=\frac{i'}{\sigma}, \quad \ldots\ldots(31)$$

where σ denotes the cross-section of the given conductor.

For this special system formulae (28) can thus be written

$$\left.\begin{aligned} a_1 &= \frac{1}{\mathfrak{V}}\int\frac{i'ds'}{\rho^3}[(y-y')\nu-(z-z')\mu]=\frac{1}{\mathfrak{V}}\int\frac{i'ds}{\rho^2}(m\nu-n\mu) \\ \text{or}\qquad a_1 &= \frac{1}{\mathfrak{V}}\int\frac{i'}{\rho^3}[(y-y')dz'-(z-z')dy'] \end{aligned}\right\}\ldots(32)$$

and similar expressions for β_1 and γ_1; a_1, β_1, γ_1 are here the components of the force that acts on unit magnetic pole.

Let us next examine the expressions (32) for a_1, β_1, γ_1. We draw two straight lines $\overline{OA}$ and $\overline{OB}$ of unit length from the origin of our system of coordinates parallel to the direction-cosines l, m, n and λ, μ, ν respectively; the coordinates of the points A and B will then be l, m, n and λ, μ, ν respectively. It is evident from the annexed figure that

$$\Delta\overline{OAB}=\frac{\sin(\rho,\, ds')}{2}$$

and $\Delta\overline{OA'B'}=\Delta\overline{OAB}\cos(n,\, z)=\dfrac{\sin(\rho,\, ds')}{2}\cos(n,\, z)$, (33)

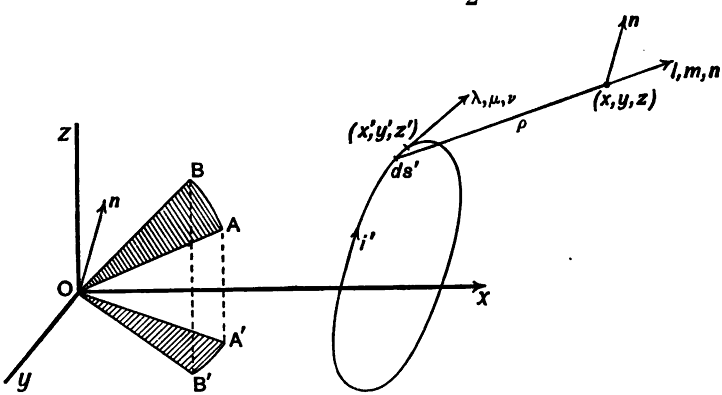

FIG. 27.

where $\Delta\overline{OA'B'}$ is the projection of $\Delta\overline{OAB}$ on the xy-coordinate-plane, and n is the normal to the latter triangular surface at O.

The area of the $\Delta \overline{OA'B'}$ is now given by the following determinate:

$$\Delta \overline{OA'B'} = \tfrac{1}{2} \begin{vmatrix} lm \\ \lambda\mu \end{vmatrix} = \tfrac{1}{2}(l\mu - m\lambda).$$

Replace $\Delta \overline{OA'B'}$ by this value in equation (33), and we get

$$\left.\begin{aligned} l\mu - m\lambda &= \sin(\rho, ds') \cos(n, z) \\ \text{and similarly} \quad m\nu - n\mu &= \sin(\rho, ds') \cos(n, x) \\ n\lambda - l\nu &= \sin(\rho, ds') \cos(n, y) \end{aligned}\right\} \quad \text{.........(34)}$$

By these relations the second expression for a_1 in formulae (32) can then be written

$$a_1 = \frac{1}{\mathfrak{B}} \int \frac{i' ds'}{\rho^2} \sin(\rho, ds') \cos(n, x); \quad \text{..........(35)}$$

β_1 and γ_1 are given by similar integrals.

The components of the magnetic force exerted by any volume-element $\sigma ds'$ on unit magnetic pole will, therefore, be

$$\left.\begin{aligned} da_1 &= \frac{i' ds' \sin(\rho, ds')}{\mathfrak{B}\rho^2} \cos(n, x), \\ d\beta_1 &= \frac{i' ds' \sin(\rho, ds')}{\mathfrak{B}\rho^2} \cos(n, y), \\ d\gamma_1 &= \frac{i' ds' \sin(\rho, ds')}{\mathfrak{B}\rho^2} \cos(n, z). \end{aligned}\right\} \quad \text{............(36)}$$

The resultant magnetic force,

$$\sqrt{da_1^2 + d\beta_1^2 + d\gamma_1^2} = \frac{i' ds' \sin(\rho, ds')}{\mathfrak{B}\rho^2}, \quad \text{........(37)}$$

will thus act not along the vector ρ defined by the two points (x', y', z') and (x, y, z) but at right angles to it along the normal n.

In order to determine the direction in which the normal n is to be drawn, suppose that the linear conductor passes through the origin O of our system of coordinates and coincides with its z-axis at that point, and that the

magnetic pole is placed at any point on its positive y-axis. If the current flows through the origin O in the direction of the positive z-axis, the normal n must then be drawn in the direction of the positive x-axis, as indicated in the annexed figure; this follows from the second of relations (34), for here $m\nu$ and $\sin(\rho, ds')$ are evidently positive and $n\mu$ vanishes, so that $\cos(n, x)$, and hence $\angle(n, x)$, must be taken positive. We see, therefore,

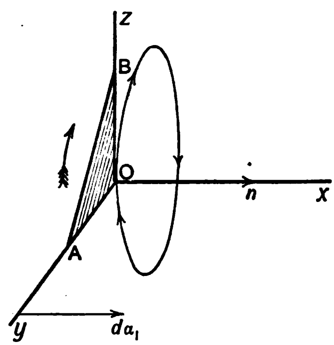

FIG. 28.

that that normal must be chosen, viewed from which the direction of rotation AB appears clockwise. Ampère has given the following practical rule for determining the direction in which this normal is to be drawn: the normal n is always to be drawn on the same side as that of the left arm of a swimmer, who, with his face directed towards the magnetic pole, advances in the direction of the current.

The assumption of the validity of equations (36) and (37), for, strictly speaking, a single volume-element cannot exert any force whatever on a magnetic pole, must form another feature of our concrete representation; we assume, namely, that every element ds' of our linear conductor (of unit cross-section) exerts on unit magnetic pole a force, whose direction is that of the normal n to

the plane that passes through ds' and the vector ρ, and whose magnitude is

$$\frac{i'ds'\sin(\rho, ds')}{\mathfrak{B}\rho^2}.$$

The desired values (35) for α_1, β_1, γ_1 follow, of course, from this assumption. We observe that the force exerted by the element ds' on unit magnetic pole is entirely independent of the medium, in which either the element itself or the magnetic pole is placed.

We have assumed in the above equations that the quantities p, q, r are measured in electrostatic units; if we employ any other system of units we must evidently write

$$p = hp_h, \quad q = hq_h, \quad r = hr_h,$$

(cf. formulae (19, III.)). The quantity of electricity that flows through unit cross-section in unit time, measured in this new system of units, must thus be written

$$i' = hi'_h.$$

Substituting this value for i' in the above expression (37) we have

$$\frac{hi'_h ds'\sin(\rho, ds')}{\mathfrak{B}\rho^2}.$$

When this expression reduces to

$$\frac{i'ds'\sin(\rho, ds')}{\rho^2},$$

the current-strength is said to be measured in magnetic units; the arbitrary constant h must then evidently be chosen equal to the given constant $\mathfrak{B}$. To obtain the so-called system of magnetic units and our fundamental equations referred to this system, we need, therefore, only to write $\mathfrak{B}$, the velocity of propagation of the electric waves in air, in place of h in all the equations at the end of § 8.

The magnetic unit of electricity E_m, divided by the electric unit E_s is thus equal to $\mathfrak{V}$, where

$$\mathfrak{V}=\frac{1}{\sqrt{\mu_a K_a}}=3\times 10^{10} \text{ cm. per sec.}$$

(cf. formula (5, II.)). It follows, therefore, that a current-strength i_s or a quantity e_s of electricity, measured in electrostatic units, divided by the same current-strength i_m or quantity e_m of electricity respectively, measured in magnetic units, will be 3×10^{10}, and that the dimensions of a quantity of electricity, measured in the latter units, will be those of a quantity of electricity, measured in the former, divided by a velocity.

To illustrate not only the use of formulae (36) and (37) but the new feature of our concrete representation, let us determine the force exerted by an infinitely long straight wire carrying an electric current on a magnetic pole A at finite distance from it. By our concrete representation and formula (37), the force df exerted by any element ds' of our wire on unit magnetic pole is

$$df=\frac{i'ds'\sin(\rho, ds')}{\rho^2},$$

where i' denotes its current-strength measured in magnetic units.

The annexed figure gives the relations

$$\rho^2=p^2+s'^2,$$

$$\sin(\rho, ds')=\sin(90^\circ+\epsilon)=p/\rho;$$

by which the above expression for df can be written

$$df=\frac{i'ds'p}{(p^2+s'^2)^{\frac{3}{2}}}.$$

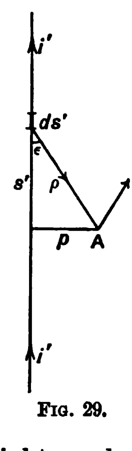

FIG. 29.

We know now that the force df acts at right angles to the plane passing through the element ds' and the

vector ρ—in the annexed figure this force is directed away from the student.

As the force exerted by every element ds' is normal to the same plane (s', ρ), the total force acting on unit magnetic pole at A will be

$$f = i'p\int_{-\infty}^{\infty} \frac{ds'}{(p^2+s'^2)^{\frac{3}{2}}},$$

which, integrated, gives

$$f = i'/p \left| \frac{s'}{\sqrt{p^2+s'^2}} \right|_{-\infty}^{+\infty} = \frac{2i'}{p},$$

where the current-strength i' is measured in magnetic units.

It follows, therefore, that a straight current of infinite length exerts on a magnetic pole at finite distance from it a force directly proportional to its current-strength and inversely proportional to its distance from the pole; compare also the problem on pp. 159-160.

CHAPTER XII.

SECTION XXVII. THE MAGNETIC FORCES GENERATED BY AN ELEMENTARY CIRCUIT; THOSE GENERATED BY A SOLENOID.

LET us determine the values of a_1, β_1, γ_1 at any point of space, to which an electric circuit of very small dimensions, lying approximately in a plane, gives rise; we shall designate such a circuit as an elementary circuit. If we grant the possibility of real magnetism, a_1, β_1, γ_1 can be regarded as the forces exerted by this circuit on unit magnetic pole. Let the point at which the magnetic pole is placed be chosen as origin of our system of coordinates and the xy-plane be laid parallel to that of our circuit. Denote the x and z-coordinates of any point O' enclosed by our circuit by $\mathfrak{p}$ and $\mathfrak{q}$ respectively; let the xy-coordinate-plane be so chosen that its y-coordinate vanishes. The coordinates of any linear element ds' of our circuit will then be

$$x' = \mathfrak{p} + \xi, \quad y' = \eta, \quad z' = \mathfrak{q}, \qquad (1)$$

where x', y', z' are referred to the origin O, and ξ, η to the point O'.

These relations, differentiated, give

$$dx' = d\xi, \quad dy' = d\eta, \quad dz' = 0. \qquad (2)$$

Lastly, let us assume that the direction of the current is that indicated in the annexed figure.

The last expressions for α_1, β_1, γ_1 in formulae (32) reduce here by formulae (1) and (2) to

$$\alpha_1=\frac{\mathfrak{q}i'}{\mathfrak{B}}\int\frac{d\eta}{\rho^3},\quad \beta_1=-\frac{\mathfrak{q}i'}{\mathfrak{B}}\int\frac{d\xi}{\rho^3},\quad \gamma_1=\frac{i'}{\mathfrak{B}}\int\frac{\eta d\xi-(\mathfrak{p}+\xi)d\eta}{\rho^3}\ldots.(3)$$

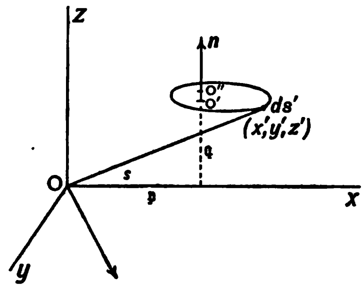

FIG. 30.

To evaluate these integrals, we first determine ρ^{-3} in terms of $\mathfrak{p}$, $\mathfrak{q}$, ξ and η. We have

$$\rho^2=x'^2+y'^2+z'^2=\mathfrak{p}^2+\mathfrak{q}^2+2\mathfrak{p}\xi+\xi^2+\eta^2,$$

or
$$\rho^2=\mathfrak{t}^2\left(1+\frac{2\mathfrak{p}\xi}{\mathfrak{t}^2}+\frac{\xi^2+\eta^2}{\mathfrak{t}^2}\right),$$

where $\mathfrak{t}^2=\mathfrak{p}^2+\mathfrak{q}^2$;

hence
$$\rho^{-3}=\mathfrak{t}^{-3}\left(1+\frac{2\mathfrak{p}\xi}{\mathfrak{t}^2}+\frac{\xi^2+\eta^2}{\mathfrak{t}^2}\right)^{-\frac{3}{2}}.$$

Upon introducing the condition that our circuit is of infinitely or very small dimensions and thus rejecting all quantities of higher infinitesimal order than the first—ξ and η are to be regarded as infinitesimals of the first and ξ^2, η^2 and $\xi\eta$ as infinitesimals of the second order in comparison to $\mathfrak{p}$ and $\mathfrak{q}$—we can develop this expression for ρ^{-3} according to ascending powers of $\mathfrak{p}\xi$, and we find

$$\rho^{-3}=\mathfrak{t}^{-3}\left(1-\frac{3\mathfrak{p}\xi}{\mathfrak{t}^2}\right).$$

Substituting this value for ρ^{-3} in the above integrals (3) for α_1, β_1, γ_1, we have

$$\left.\begin{aligned}\alpha_1=\frac{\mathfrak{q}i'}{\mathfrak{R}\mathfrak{t}^3}\int d\eta-\frac{3\mathfrak{p}\mathfrak{q}i'}{\mathfrak{R}\mathfrak{t}^5}\int\xi d\eta,\quad \beta_1=-\frac{\mathfrak{q}i'}{\mathfrak{R}\mathfrak{t}^3}\int d\xi+\frac{3\mathfrak{p}\mathfrak{q}i'}{\mathfrak{R}\mathfrak{t}^5}\int\xi d\xi,\\ \gamma_1=-\frac{\mathfrak{p}i'}{\mathfrak{R}\mathfrak{t}^3}\int d\eta+\frac{3\mathfrak{p}^2i'}{\mathfrak{R}\mathfrak{t}^5}\int\xi d\eta+\frac{i'}{\mathfrak{R}\mathfrak{t}^3}\int(\eta d\xi-\xi d\eta);\end{aligned}\right\}\quad(4)$$

in the expression for γ_1 we have retained quantities of the first order of magnitude only. As all the given integrations are to be extended round the given circuit from any given point back to that point, the following integrals will evidently vanish:

$$\int d\eta,\quad \int d\xi \quad\text{and}\quad \int\xi d\xi.$$

To evaluate the remaining integrals, we observe that the area of the triangular surface-element df, whose sides are formed by the element ds' and the vectors from the point O' to the ends of this element, is given by the determinate

$$df=\tfrac{1}{2}\begin{vmatrix}\xi, & \eta,\\ \xi+d\xi, & \eta+d\eta\end{vmatrix}=\tfrac{1}{2}(\xi d\eta-\eta d\xi),$$

which, integrated, gives

$$f=\tfrac{1}{2}\int(\xi d\eta-\eta d\xi);\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

that is, the value of this integral is $2f$, where f denotes the surface enclosed by our circuit. The values of the remaining integrals follow directly from the integral-equation

$$\int(\xi d\eta+\eta d\xi)=\int d(\xi\eta)=0$$

and formula (5) $\int(\xi d\eta-\eta d\xi)=2f$,

which give $\int\xi d\eta=f$ and $\int\eta d\xi=-f$.

Formulae (4) thus reduce to

$$\left.\begin{aligned}\alpha_1 &= -\frac{3\mathfrak{p}\mathfrak{q}i'}{\mathfrak{V}\mathfrak{t}^5}f, \quad \beta_1 = 0,\\ \gamma_1 &= \frac{i'f}{\mathfrak{V}}\left(\frac{3\mathfrak{p}}{\mathfrak{t}^5} - \frac{2}{\mathfrak{t}^3}\right) = \frac{i'f}{\mathfrak{V}}\frac{d}{d\mathfrak{q}}\left(\frac{\mathfrak{q}}{\mathfrak{t}^3}\right)\end{aligned}\right\} \quad \text{.............. (6)}$$

These equations can be interpreted mechanically. We erect at the point O' that normal to the surface f, from which the direction of the current appears clockwise, as indicated in figure 30, and lay off on this normal an infinitely short distance $O'O'' = \delta$; we then imagine at each point O' and O'' a mass

$$m = \frac{i'f}{\mathfrak{V}\delta}, \quad \text{.......................... (7)}$$

which acts on unit mass at the origin O, the former with a repulsive and the latter with an attractive force m times the inverse square of the distance. The components X', Y', Z' of the force exerted by the mass m at the point O' on unit mass at the origin O will evidently be

$$X' = -\frac{m\mathfrak{p}}{\mathfrak{t}^3}, \qquad Y' = 0, \quad Z' = -\frac{m\mathfrak{q}}{\mathfrak{t}^3},$$

and those, X'', Y'', Z'', by the mass m at the point O''

$$X'' = -X' - \delta\frac{dX'}{d\mathfrak{q}}, \quad Y'' = 0, \quad Z'' = -Z' - \delta\frac{dZ'}{d\mathfrak{q}};$$

the components of the total resultant force acting at the origin will thus be

$$X' + X'' = -\delta\frac{dX'}{d\mathfrak{q}}, \quad Y' + Y'' = 0, \quad Z' + Z'' = -\delta\frac{dZ'}{d\mathfrak{q}};$$

or, since $$\frac{dX'}{d\mathfrak{q}} = \frac{dX'}{d\mathfrak{t}}\cdot\frac{d\mathfrak{t}}{d\mathfrak{q}} = \frac{3m\mathfrak{p}\mathfrak{q}}{\mathfrak{t}^5}$$

and
$$\frac{dZ'}{d\mathfrak{q}}=\frac{3m\mathfrak{q}^2}{\mathfrak{t}^5}-\frac{m}{\mathfrak{t}^3},$$

$$X'+X''=-\frac{3m\mathfrak{p}\mathfrak{q}\delta}{\mathfrak{t}^5},\quad Y'+Y''=0,\quad Z'+Z''=\frac{m\delta}{\mathfrak{t}^3}-\frac{3m\mathfrak{q}^2\delta}{\mathfrak{t}^5},$$

or by formula (7)
$$X'+X''=-\frac{3\mathfrak{p}\mathfrak{q}i'}{\mathfrak{P}\mathfrak{t}^5}f,\quad Y'+Y''=0,\quad Z'+Z''=\frac{i'f}{\mathfrak{P}\mathfrak{t}^3}-\frac{3\mathfrak{q}^2i'}{\mathfrak{P}\mathfrak{t}^5}f,$$

which are identical to the expressions already found for α_1, β_1, γ_1 on the preceding page.

The above expressions (6) for α_1, β_1, γ_1 are, of course, entirely independent of the position of our system of coordinates; the special system we have chosen above corresponds only to a transformation of coordinates. In general, we shall have, therefore,
$$\alpha_1=X'+X'',\quad \beta_1=Y'+Y'',\quad \gamma_1=Z'+Z'',$$

We know now from the theory of the potential that
$$X'+X''=\frac{d\psi}{dx},\quad Y'+Y''=\frac{d\psi}{dy},\quad Z'+Z''=\frac{d\psi}{dz},$$

where ψ is the (Newtonian) potential of the given masses; here ψ has the value
$$\psi=m\left(\frac{1}{\overline{O''M}}-\frac{1}{\overline{O'M}}\right),$$

where M denotes the point (x, y, z), at which the potential is sought. We can thus write
$$\left.\begin{aligned}\alpha_1&=\frac{d\psi}{dx}=\frac{d}{dx}m\left(\frac{1}{\overline{O''M}}-\frac{1}{\overline{O'M}}\right),\\ \beta_1&=\frac{d\psi}{dy}=\frac{d}{dy}m\left(\frac{1}{\overline{O''M}}-\frac{1}{\overline{O'M}}\right),\\ \gamma_1&=\frac{d\psi}{dz}=\frac{d}{dz}m\left(\frac{1}{\overline{O''M}}-\frac{1}{\overline{O'M}}\right),\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots(8)$$

Hereby we have won a mechanical interpretation for the magnetic forces α_1, β_1, γ_1, which we can introduce into our concrete representation as a new feature; we conceive, namely, that the components of the force exerted by an elementary circuit on unit magnetic pole at any point of space are the partial derivatives with regard to the coordinates of a function ψ, which is the potential of two masses $m=\frac{i'f}{\mathfrak{B}\delta}$ placed at the points O' and O'' as specified above and acting on unit mass, the one with the attractive and the other with the repulsive force m times the inverse square of the distance. Such a system can be realized by placing a magnet of length δ and moment $\frac{i'f}{\mathfrak{B}}$ with its one pole at the point O' and its other at the point O''; the components $\alpha_1, \beta_1, \gamma_1$ of the magnetic force generated by an elementary circuit could thus always be produced by replacing the given circuit by a small but sufficiently powerful magnet.

To obtain a geometrical interpretation for ψ we describe a sphere of radius $\overline{MO''}=\rho'$ about the point M, intersecting the straight line that passes through M and O' at the point A. We have then (cf. figure 31, next page)

$$\rho'=\overline{MA}=\overline{MO'}+\overline{O'A}=\rho+\delta\cos\epsilon,$$

where ϵ denotes the angle between the vectors $\overline{O'A}$ and $\overline{O'O''}$ (δ).

We can thus write the above expression (8) for ψ in the form

$$\psi=m\left(\frac{1}{\rho+\delta\cos\epsilon}-\frac{1}{\rho}\right).$$

As δ is small in comparison to both the vector ρ and the angle ϵ, we can develop $\frac{1}{\rho+\delta\cos\epsilon}$ according to ascending-powers of δ, namely

$$\frac{1}{\rho+\delta\cos\epsilon}=\frac{1}{\rho}-\frac{\delta\cos\epsilon}{\rho^2}+\dots,$$

and thus write ψ as follows:

$$\psi=-\frac{m\delta\cos\epsilon}{\rho^2}=-\frac{i'f\cos\epsilon}{\mathfrak{V}\rho^2}.\quad\dots\dots\dots\dots(9)$$

From the point M we next draw tangents to our elementary circuit; if f' denotes the area of the plane drawn through the point O' at right angles to the vector

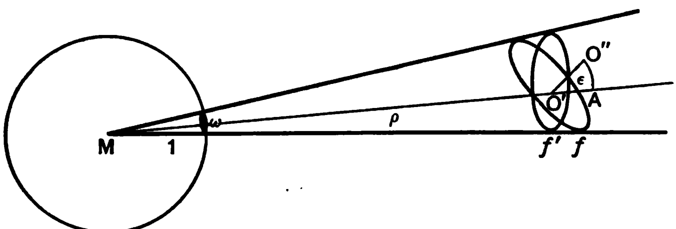

FIG. 31.

ρ and intercepted by the cone formed by these tangents, it is evident, since these tangents are approximately parallel to one another, that

$$f'=f\cos(f,f')=f\cos\epsilon.\quad\dots\dots\dots\dots(10)$$

Lastly, we describe a second sphere of unit radius about the point M, and we get

$$\omega:f'=1:\rho^2,$$

where ω denotes the area of that part of the sphere intercepted by the cone. These last two relations give

$$\omega=\frac{f\cos\epsilon}{\rho^2};$$

by which the last expression (9) for ψ can be written

$$\psi=-\frac{i'\omega}{\mathfrak{V}};\quad\dots\dots\dots\dots(11)$$

ω is the solid angle subtended by the elementary circuit at the point M. When the current viewed from this point flows in the contraclockwise direction, ψ will evidently be negative (cf. formula (8) and figure 30), and the solid angle ω must thus be taken positive. We could, indeed, designate ω as the number of rays that pass through the surface f from the side, viewed from which the current flows in the contraclockwise direction, to the opposite side.

In our concrete representation we could thus conceive the quantity ψ as $\left(-\frac{i'}{\mathfrak{V}}\right)$ times the solid angle ω subtended by our elementary circuit at the point (M), in which the values of α_1, β_1, γ_1 are sought. This geometrical analogy often proves useful, as we shall see directly.

A number of elementary circuits placed at equal distances apart along any line with their planes at right angles to that line is called a solenoid. To find the components α_1, β_1, γ_1 of the magnetic force exerted by a solenoid on unit magnetic pole at any point M, we make use of the first new feature of our concrete representation and conceive that every elementary circuit of the solenoid is replaced by two (magnetic) masses $m=\frac{i'f}{\mathfrak{V}\delta}=\frac{i'fN}{\mathfrak{V}}$—we put $\delta=1/N$, where N denotes the number of elementary circuits per unit length, and that the one attracts and the other repels the given (magnetic) mass at M; the potential ψ of these (magnetic) masses is then the potential sought, that, namely, whose derivatives with regard to the coordinates give the desired values for the forces $\alpha_1, \beta_1, \gamma_1$. We thus have

$$\psi=m\left[\left(\frac{1}{\overline{O^{n+1}M}}-\frac{1}{\overline{O^{n}M}}\right)+\left(\frac{1}{\overline{O^{n}M}}-\frac{1}{\overline{O^{n-1}M}}\right)+\ldots\left(\frac{1}{\overline{O''M}}-\frac{1}{\overline{O'M}}\right)\right],$$

where n denotes the total number of elementary circuits, hence

$$\psi = m\left(\frac{1}{O^{n+1}M} - \frac{1}{O'M}\right). \quad\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

We see, therefore, that we can always conceive the magnetic forces α_1, β_1, γ_1, to which a solenoid gives rise, as due to the potential of only two of these (magnetic) masses, namely, of those at its two ends, the points O' and O^{n+1}; if we conceive the masses as magnetic masses, the mass at the point O' must be taken positive and that at the point O^{n+1} negative in the given case. A magnet of length $\frac{n+1}{N}$ and moment $\frac{(n+1)i'f}{\mathfrak{B}}$, with its positive end at O' and its negative end at O^{n+1}, would thus produce the same magnetic field as the given solenoid. In general, we must conceive the positive magnetic mass m at that end of the solenoid, viewed from which the current flows in the contraclockwise direction.

It is customary to call that end of a magnet which points north its north end, and that which points south its south end. Boreal magnetism is that which is supposed to exist near the north pole of the earth or the south end of a magnet and austral magnetism that which belongs to the south pole of the earth or the north end of a magnet. The direction of the magnetic force is taken as that in which austral magnetism tends to flow, that is, from south to north, this being the positive direction of the lines of magnetic force. It is, moreover, customary to designate austral magnetism or that in the north end of a magnet as positive, and boreal magnetism or that in the south end of a magnet as negative. We shall retain these conventions here, although they are not quite consistent with the above, for we have always denoted that side of the current, viewed from which the current flows in the clockwise direction, as positive. Lastly, we should not forget that the (magnetic) masses

m introduced above do not actually exist, but merely form a feature of our concrete representation.

All the above formulae hold not only for $\eta_r \gtrless 0$ but for $\eta_r = 0$; we should, however, observe that in the latter case the physical meaning of the quantities α_1, β_1, γ_1 has yet to be found—we shall return to this subject in Chapter XIV.

Ampère's hypothesis only assumes the existence of elementary or molecular currents; how these molecular currents are generated does not of course follow from Maxwell's equations. According to Ampère's hypothesis, a body is said to show no signs of magnetization until external forces have been brought to act on the molecular currents within it; for, as long as no external forces act on the body, the molecular currents will assume no definite arrangement, that is, no plane among the infinite number of planes, in which these currents lie, will predominate, and, consequently, no magnetic effects will be manifested. On the other hand, a certain plane is supposed to predominate in the magnetic body; the currents that lie in this predominating plane form solenoids, and these solenoids produce the desired magnetic effects, as has just been shown above. This solenoidal arrangement of the molecular currents can be brought about by the approach of a magnet to the given body; the nearer the magnet is brought to the body, the greater the number of currents turned into the predominating plane, and hence the more pronounced the magnetic effects exhibited by it.

If we assume the validity of the principles of the preservation of the centres of gravity and areas of ponderable bodies irrespective of the presence of an ether, that is, if we assume that no appreciable progressive or rotatory momentum is imparted to the ether itself, the apparent action and reaction between bodies at a distance must then be the same in all media (vacuum or air), that have no appreciable effect on the action of electric and magnetic forces. It thus follows that,

although we may not be able to maintain that every magnetic pole m' exerts on every current-element a force equal but opposite in direction to m' times the expression (37, XI.), the total force exerted by any such magnetic pole—provided real magnetism exists—on any closed circuit will be given by the expression obtained on this assumption. Similarly, the magnetic pole m' at any point (x, y, z) will exert on the positive end of a solenoid at any point (x', y', z') a force, whose components are

$$\frac{m'i'fN(x-x')}{\mathfrak{B}\rho^3}, \quad \frac{m'i'fN(y-y')}{\mathfrak{B}\rho^3}, \quad \frac{m'i'fN(z-z')}{\mathfrak{B}\rho^3},$$

where ρ denotes the distance between these two points. This force could also be brought to act on the given pole by replacing the solenoid end by a magnetic pole of the strength

$$\frac{m'i'fN}{\mathfrak{B}} = m'i'_m fN;$$

here i' is the current-strength measured in electrostatic units, and i'_m the same measured in magnetic units.

SECTION XXVIII. DETERMINATION OF THE MAGNETIC FORCES GENERATED BY ANY CIRCUIT; WORK DONE BY THESE FORCES. THE MAGNETIC ENERGY OF THE FIELD.

To determine the values of $\alpha_1, \beta_1, \gamma_1$ due to the presence of any closed circuit of current-strength i' (measured in electrostatic units), we imagine that the given circuit is the boundary of any surface F, divide the latter into an infinite number of infinitely small surface-elements and conceive that the edge of every element is traversed by a current of the same current-strength i' and direction of flow as the main current; this network of elementary circuits will then produce the same effects as our given

circuit, since the elementary currents within the imaginary surface F all cancel one another, whereas those flowing along its boundary unite in forming the main circuit itself. We can therefore conceive the given circuit as replaced by the net-work of elementary circuits and thus examine the latter instead of the former. In conformity to our concrete representation, we can imagine that every elementary circuit is replaced by a magnet, whose north pole lies in the respective surface-element of the surface F, and whose south pole is at the distance δ from that element on its positive normal, that is, that the elementary circuits that replace the main circuit are replaced themselves by two surfaces, the arbitrary surface F and its concentric surface F' at the distance δ from it, covered uniformly with magnetism of the surface-density $\frac{i'}{\mathfrak{B}\delta}$, the former with positive or austral and the latter with negative or boreal magnetism. We know then that

$$\alpha_1 = \frac{d\psi}{dx}, \quad \beta_1 = \frac{d\psi}{dy}, \quad \gamma_1 = \frac{d\psi}{dz},$$

where ψ is the potential of the magnetism on both surfaces; or we can define $-\psi$ as the solid angle

$$\Omega = \Sigma\omega, \qquad \text{.........................(13)}$$

subtended by the main circuit at the point, where the values of $\alpha_1, \beta_1, \gamma_1$, are sought, times the current-strength $i'_m = i'/\mathfrak{B}$. Ω, as ω above, is to be taken positive, when the current viewed from the given point flows in the contraclockwise direction, and negative, when in the clockwise direction, provided Ω is smaller than 2π; this condition can be omitted, if we assume that Ω becomes discontinuous as the solenoid pole passes through the given circuit from its negative, where $\Omega = 2\pi$, to its positive side, where Ω shall assume the value -2π, decreasing in absolute value as the solenoid pole recedes.

By this assumption or rather convention, Ω also becomes single-valued—it would otherwise be multiple-valued,

$$\Omega = \Omega_0 + 4\pi m,$$

where m denotes the number of times the solenoid pole has passed through the circuit from its negative to its positive side and Ω_0 the initial $(m=0)$ value of Ω at the given point.

Let us determine the values of α, β, γ due to the presence of an infinitely long straight wire carrying a current of strength i'. We shall employ here the above geometrical interpretation for ψ. We lay our xy-coordinate-plane through the point M, at which the values of $\alpha_1, \beta_1, \gamma_1$, are sought, at right angles to the direction of the wire, as indicated in figure 32 below, describe a sphere of unit radius about this point and draw tangents thence to the given circuit. The surface-area intercepted by these tangents on the sphere will be Ω, the solid angle sub-

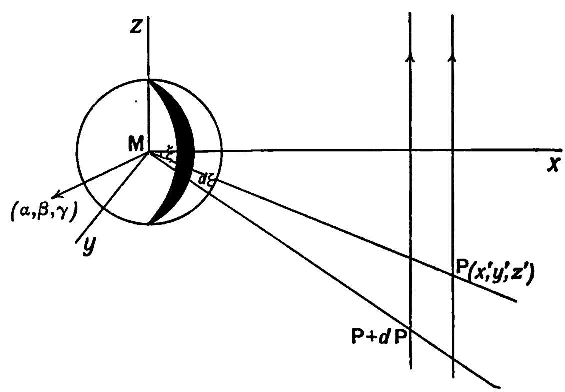

FIG. 32.

tended by the circuit at the point M. As here the tangents intercept the sphere in a meridian or line only, we can speak of neither the surface nor the solid angle Ω; it is, however, the variation of Ω and not Ω itself

that is sought. To determine the former we vary the position of the wire; let a variation dP in the position of the wire in the xy-plane, corresponding to a variation $d\xi$ in the angle ξ between the x-axis and the vector $\overline{MP}$, cause a variation $d\Omega$ in Ω; $d\Omega$ is evidently the cyclical surface intercepted on the sphere by the tangents from the point M to the given and varied positions of the wire. The figure gives the relation

$$d\Omega : d\xi = 4\pi : 2\pi,$$

or

$$d\Omega = 2d\xi,$$

hence

$$\frac{d\Omega}{dx'} = 2\frac{d\xi}{dx'},$$

$$\frac{d\Omega}{dy'} = 2\frac{d\xi}{dy'}.$$

We thus find

$$a_1 = \frac{d\psi}{dx} = -\frac{i'}{\mathfrak{B}}\,\frac{d\Omega}{dx} = -\frac{i'}{\mathfrak{B}}\,\frac{d\Omega}{dx'}\,\frac{dx'}{dx} = \frac{2i'}{\mathfrak{B}}\,\frac{d\xi}{dx'},$$

$$\beta_1 = \frac{2i'}{\mathfrak{B}}\,\frac{d\xi}{dy'}, \quad \gamma_1 = \frac{2i'}{\mathfrak{B}}\,\frac{d\xi}{dz'} = 0.$$

If the vector $\overline{MP}$ coincides with the x-axis, these values reduce to

$$a_1 = \gamma_1 = 0, \quad \beta_1 = \frac{2i'}{\mathfrak{B}}\,\frac{d\xi}{dy'} = \frac{2i'}{\mathfrak{B}}\Big/ p;$$

compare the value for f on p. 246.

The formulae of this article still remain valid, when $\eta_r = 0$. If we accept the existence of real magnetism, the component along the x-axis of the force that acts on the quantity m_r of real magnetism will be $m_r \,.\, a$. As the number of lines of force that radiate from the magnetic pole m_r is $4\pi m_r$ by formula (17, XI.), we thus have

$$\Omega = \frac{Z}{m_r},$$

where Z denotes the number of lines of force which the pole m_r sends through the given circuit from the side viewed from which the current flows in the contraclockwise direction. We can therefore write the expression (11) for ψ as follows:

$$\psi = -\frac{i'Z}{m_r\mathfrak{B}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

the components X, Y, Z of the magnetic force that acts on the quantity m_r of real magnetism can thus be written

$$\left.\begin{aligned} X &= m_r\frac{d\psi}{dx} = -\frac{i'}{\mathfrak{B}}\frac{dZ}{dx} = -i'_m\frac{dZ}{dx} \\ Y &= m_r\frac{d\psi}{dy} = -\frac{i'}{\mathfrak{B}}\frac{dZ}{dy} = -i'_m\frac{dZ}{dy} \\ Z &= m_r\frac{d\psi}{dz} = -\frac{i'}{\mathfrak{B}}\frac{dZ}{dz} = -i'_m\frac{dZ}{dz} \end{aligned}\right\} \quad \ldots\ldots\ldots(15)$$

The work done by these forces, which act apparently at a distance between the magnetic pole m_r and the given circuit, during any displacement δx, δy, δz of the former is evidently

$$\begin{aligned} X\delta x + Y\delta y + Z\delta z &= -i'_m\left(\frac{dZ}{dx}\delta x + \frac{dZ}{dy}\delta y + \frac{dZ}{dz}\delta z\right) \\ &= -i'_m\delta Z. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16) \end{aligned}$$

If we have n magnetic poles, $m_{r,1}$, $m_{r,2}$, ... $m_{r,n}$, instead of only the one m_r, the work done during their respective displacements δx_h, δy_h, δz_h, $h = 1, 2, \ldots n$, will be

$$\sum_{h=1}^{h=n}(X_h\delta x_h + Y_h\delta y_h + Z_h\delta z_h) = -i'_m\Sigma\delta Z = -i'_m\delta\Sigma Z, \quad (17)$$

where ΣZ is the total number of lines of force sent through the given circuit by the given poles.

On the other hand, it follows from the principles of the preservation of the centres of gravity and areas of

ponderable bodies (cf. text, p. 256) that the work done during any displacement δx, δy, δz of the circuit by the forces that act apparently at a distance between it and the magnetic poles must be

$$\sum_{h=1}^{h=n}(X_h\delta x+Y_h\delta y+Z_h\delta z)$$

$$=-\sum_{h=1}^{h=n}i'_m\left(\frac{dZ_h}{dx}\delta x+\frac{dZ_h}{dy}\delta y+\frac{dZ_h}{dz}\delta z\right)$$

$$=-i'_m\sum_{h=1}^{h=n}\delta Z=-i'_m\delta\sum_{h=1}^{h=n}Z, \text{ ....} (18)$$

which is identical with expression (17).

A deformation of the main circuit can be brought about by displacements of its elementary circuits, since we can always conceive that the former is replaced by the latter. The work done during any deformation of the main circuit by the forces that act between it and the magnetic poles of the field can therefore be found by determining the work done during the corresponding displacements of the elementary circuits by the forces that act between them and the given poles; this work can evidently be expressed as follows:

$$\sum_{h=1}^{h=n}\sum_{k=1}^{k=\nu}(X_{h,k}\delta x_k+Y_{h,k}\delta y_k+Z_{h,k}\delta z_k)$$

$$=-\sum_{h=1}^{h=n}\sum_{k=1}^{k=\nu}i'_m\left(\frac{d\zeta_{h,k}}{dx}\delta x_k+\frac{d\zeta_{h,k}}{dy}\delta y_k+\frac{d\zeta_{h,k}}{dz}\delta z_k\right)$$

$$=-\sum_{h=1}^{h=n}\sum_{k=1}^{k=\nu}i'_m\delta\zeta_{h,k}, \text{ ....} (19)$$

where ν denotes the number of elementary circuits displaced and $\zeta_{h,k}$ the number of lines of force sent through the elementary circuit k by the magnetic pole $m_{r,k}$. The expressions (18) and (19) will also represent

the work done by the forces that act apparently at a distance, when the lines of force arise from other currents, since the forces that act on any circuit depend only upon the state of the ether in its immediate neighbourhood and not upon the manner in which that state has been brought about (cf. also text, p. 239).

The number of lines of force that pass through any surface-element do of any finite surface S, whose boundary is any given circuit s, will evidently be

$$M\sqrt{a^2+\beta^2+\gamma^2}\cos(\sqrt{a^2+\beta^2+\gamma^2}, n)do$$
$$=M[a\cos(n, x)+\beta\cos(n, y)+\gamma\cos(n, z)]do,$$

where n is the positive normal to the given surface-element do, that is, that normal to the surface viewed from which the current flows in the contraclockwise direction; the so-defined positive normal thus always makes an acute angle with the direction of the lines of force. The total number of lines of force that pass through the surface S will thus be

$$\Sigma Z=\int M\,do[a\cos(n, x)+\beta\cos(n, y)+\gamma\cos(n, z)]; \quad \ldots(20)$$

the validity of this expression is of course entirely independent of any assumption concerning the existence of real magnetism.

Let us next examine the expression for the (magnetic) energy of the field due to the presence either of electric currents only or of both electric currents and real magnetism, the existence of the latter being taken for granted. For this purpose we bring a linear circuit s of current-strength i into the given field; if we denote the components of the magnetic force at any point (x, y, z) of the field by a, β, γ and those arising from the circuit s by a_1, β_1, γ_1, we know then that

$$a_1=\frac{d\psi}{dx}, \quad \beta_1=\frac{d\psi}{dy}, \quad \gamma_1=\frac{d\psi}{dz}, \quad \ldots\ldots\ldots\ldots(21)$$

where $\psi = -i_m\Omega$, Ω being the solid angle subtended by the circuit s at the point (x, y, z).

The total energy V of the field after the insertion of the circuit s is evidently given by the expression

$$V = \int \frac{M}{8\pi}[(\alpha+\alpha_1)^2+(\beta+\beta_1)^2+(\gamma+\gamma_1)^2]d\tau$$
$$= \int \frac{M}{8\pi}(\alpha^2+\beta^2+\gamma^2)d\tau + \int \frac{M}{8\pi}(\alpha_1{}^2+\beta_1{}^2+\gamma_1{}^2)d\tau$$
$$+ \int \frac{M}{4\pi}(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)d\tau$$

(cf. formula (8, II.)). The first integral of this last expression for V represents the total energy of the field, the second that of the circuit s before its insertion into the field and the third that arising from the combined presence of both the bodies constituting the field and the circuit s—this last integral does not, of course, appear, if either the field or the circuit s is wanting. The third integral could be called the mutual-potential of the field and the circuit, since the forces that act between them follow directly from it. As long as the field is kept constant, the first integral will also remain constant; the value of the second integral can only be altered either by a change in the current-strength i or by a deformation of the circuit s—a displacement of the circuit in the field will not affect its value in any way; on the other hand, the value of the third integral will be altered by any displacement of the circuit s in the field. Let us examine this last integral—we denote it by V_{12}—more carefully. We replace $\alpha_1, \beta_1, \gamma_1$ by their values (21), and we have

$$V_{12} = \int \frac{M}{4\pi}\left(\alpha\frac{d\psi}{dx}+\beta\frac{d\psi}{dy}+\gamma\frac{d\psi}{dz}\right)d\tau. \quad \text{.........(22)}$$

We have seen on pp. 258-259 that the function ψ is, in general, multiple-valued, but that it can be made single-valued by intersecting the given region by any

finite surface S, whose boundary is the given circuit (s), and by assuming that $\psi = -i_m\Omega$ becomes discontinuous on this surface, Ω making a jump of 4π as it passes through it from the side of its positive to that of its negative normal—by its positive normal we always refer to that normal to the surface viewed from which the current flows in the contraclockwise direction. By this convention ψ becomes a single-valued function and a partial integration of the above expression for V_{12} can thus be performed. Integrating by parts we have then

$$V_{12} = \frac{1}{4\pi}\iint M(\alpha\, dy\, dz + \beta\, dx\, dz + \gamma\, dx\, dy)[\psi]$$

$$-\frac{1}{4\pi}\int \psi\left[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\right] d\tau. \quad \ldots\ldots(23)$$

By the above convention ψ vanishes at both $-\infty$ and $+\infty$; at all intermediate points it is continuous, except on the surface S, where it becomes discontinuous, making a jump of $4\pi i_m$ as it passes through it in the direction of the lines of force; the value of $[\psi]$ will therefore be determined by this discontinuity alone. As the integration is to be taken in the direction of the positive normal, we have

$$[\psi] = -i_m[\Omega] = -4\pi i_m.$$

Replacing $[\psi]$ and ψ by their respective values in equation (23), we get

$$V_{12} = -i_m\iint M(\alpha\, dy\, dz + \beta\, dx\, dz + \gamma\, dx\, dy)$$

$$+\frac{i_m}{4\pi}\int \Omega\left[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\right] d\tau. \quad \ldots\ldots\ldots(24)$$

By the relations

$$dy\, dz = do \cos(n, x), \quad dx\, dz = do \cos(n, y),$$

$$dx\, dy = do \cos(n, z),$$

the first of these integrals can be written,

$$\iint M(\alpha dydz+\beta dxdz+\gamma dxdy)$$

$$=\int Mdo[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]=\Sigma Z \quad ...(25)$$

(cf. formula (20)); ΣZ is the number of lines of force sent by the field through the surface S in the direction of its positive normal. If the field is produced by real magnetism only, the quantity of real magnetism in every volume-element will be

$$\frac{d\tau}{4\pi}\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]$$

and hence the number of lines of force emitted from it

$$d\tau\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right];$$

the total number of lines of force sent by the real magnetism of the field through the surface S in the direction of its positive normal will therefore be given by the following integral:

$$\frac{1}{4\pi}\int\Omega\, d\tau\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right].$$

In the given case, where the only electric current present is that brought into the field, this integral-expression must be equal to ΣZ, that is,

$$\int Mdo[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]$$

$$=\frac{1}{4\pi}\int\Omega\, d\tau\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right].$$

From this and formulae (22) and (24) it follows then that

$$V_{12}=0,$$

that is, the total magnetic energy of any field produced by the presence of real magnetism alone will be increased by bringing an electric circuit into it by the energy of that circuit only; in other words, the total magnetic energy of a field consisting of an electric circuit and real magnetism will remain constant, as the circuit is moved through the field, provided its current-strength i is maintained constant. The work done during any displacement of the given circuit cannot therefore be derived from the energy of the medium but must be assigned to the head of energy that drives its current.

On the other hand, if no real magnetism is present, we have

$$V_{12} = -i_m \int M\,do[\alpha\cos(n, x) + \beta\cos(n, y) + \gamma\cos(n, z)]$$

$$= -i_m \Sigma Z \quad \text{...} (26)$$

(cf. formula (25)).

The work done by the forces, which an electric circuit apparently exerts on magnetic poles during any displacement of the latter, is by formula (17) equal to the corresponding diminution in the quantity $i_m \Sigma Z$; conversely this quantity $(-i_m \delta \Sigma Z)$ will thus represent the work done by the forces, which the field apparently exerts on the electric circuit during any displacement of the latter; it is also equal to the increment of the visible kinetic energy of the system (circuit). We have often observed that the work done is entirely independent of the manner in which the field has been produced, whether by real magnetism or by other electric circuits; it follows therefore from formula (19) that in any field produced by electric currents only the energy of the medium will increase by this same quantity $(-i_m \delta \Sigma Z)$ during any displacement of an electric circuit within it. Hence twice as much energy must be expended in the present case as in the preceding one; half of this energy will appear in the form of visible

energy as in the first case, whereas the other half will be transformed into invisible medium or ether energy. In both cases, however, this energy must be derived from the electromotive forces that drive the currents.

As an illustration of the above, take two circular linear circuits lying in parallel planes. If their directions of flow are similar, by diminishing the distance between them we increase the absolute value of ΣZ and hence decrease its real value, since the lines of force pass from the one circuit through the other in the negative direction; it follows therefore from formula (26) that the energy of the medium will be increased. The visible kinetic energy of the system will also be increased, since the circuits approach each other with an acceleration. We know now that these energies are derived from the electromotive forces that drive the currents. It follows, therefore, that, if their current-strengths are maintained constant and all other visible motions are excluded, the electromotive forces will have to supply no other sources of energy except the two just mentioned and Joule's heat. It thus becomes evident that the determination of the ponderable forces from the energy of the medium is only permissible, when the head of energy that drives the currents is alone present.

CHAPTER XIII.

SECTION XXIX. STOKES' THEOREM: ITS APPLICATION TO ELECTRO-MAGNETISM.

WE can state Stokes' theorem as follows: let S be any finite surface, whose contour is determined by any given closed curve s, and let u, v, w denote the components of any vector, whose value is given at every point of space, do any element of the surface S and l, m, n the direction-cosines of its normal; then

$$\int(udx+vdy+wdz)$$
$$=\int do\left[l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right], \ldots (1)$$

where the linear integral is to be extended round the curve s and the surface integral over the surface S. This theorem has been demonstrated in several ways; an analytical proof of it is given in Maxwell's treatise on *Electricity and Magnetism*, § 24, and a synthetical proof in Sir William Thomson's paper on *Vortex Motion* § 69 (q); that indicated in Thomson's and Tait's *Natural Philosophy* is similar to the original proof given by Stokes in his Smith's prize paper of 1854.

We first prove Stokes' theorem for the special case, where the curve s and the surface S lie in a plane and are of infinitely or very small dimensions. If we lay our xy-coordinate-plane parallel to that of the given surface, we have

$$l=m=0, \quad n=1, \quad dz=0,$$

and Stokes' theorem (1) thus reduces to

$$\int(udx+vdy)=\int do\left(\frac{dv}{dx}-\frac{du}{dy}\right).$$

As the surface S is assumed to be very small, the quantities $\frac{dv}{dx}$ and $\frac{du}{dy}$ can be regarded as approximately constant along it, and we have

$$\int(udx+vdy)=\left(\frac{dv}{dx}-\frac{du}{dy}\right)\int do=\left(\frac{dv}{dx}-\frac{du}{dy}\right)\omega, \quad \ldots\ldots(2)$$

where ω denotes the area of the given surface (S).

To prove the relation (2), we write its first integral in the form

$$\int(udx+vdy)=\int[(u_1-u_2)dx-(v_1-v_2)dy], \quad \ldots\ldots(3)$$

where u_1 and u_2 denote the values of u for any given value of x and the two respective values of y, and v_1 and v_2 the values of v for any given value of y and the two respective values of x. As the surface S has been assumed to be infinitely small, u_2 and v_2 can be developed by Taylor's theorem as functions of u_1 and v_1 respectively, namely,

$$u_2=u_1(x,\, y+\eta)=u_1+\eta\frac{du_1}{dy},$$

$$v_2=v_1(x+\xi,\, y)=v_1+\xi\frac{dv_1}{dx},$$

where ξ and η denote the differences between the co-ordinates x_1 and x_2 and y_1 and y_2 respectively. Substituting these values for u_2 and v_2 in the above integral (3), we have

$$\int(udx+vdy)=\int\left(-\eta\frac{du}{dy}dx+\xi\frac{dv}{dx}dy\right),$$

or, since $\frac{du}{dy}$ and $\frac{dv}{dx}$ can be regarded as approximately constant along the given surface

$$\int(u\,dx+v\,dy)=\left(\frac{dv}{dx}-\frac{du}{dy}\right)\omega, \quad\ldots\ldots\ldots\ldots\ldots(4)$$

the special form of Stokes' theorem for the given case (cf. formula (2)); analogous relations hold for similar surfaces parallel to the yz- and xz-coordinate-planes.

To obtain a general proof of Stokes' theorem, where namely S is any finite surface, we divide the given surface into an infinite number of infinitely small triangular surfaces by laying three systems of planes

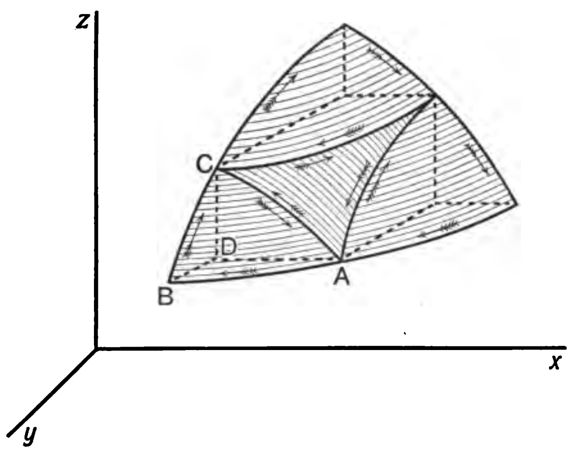

FIG. 33.

through it parallel to the coordinate-planes—four such triangular surfaces are graphically represented in the annexed figure—and we first evaluate the integral

$$\int(u\,dx+v\,dy+w\,dz), \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

extended over any one of these surface-elements, as ABC of figure 33. For this purpose we project the given triangular surface-element ABC on the three planes, by which it is intercepted; its projections ADB, BDC, ADC, where D is the point of intersection of the three given planes, are

$$ADB = n\omega, \quad BDC = l\omega, \quad ADC = m\omega,$$

where ω denotes the area of the surface-element ABC and l, m, n the direction-cosines of its normal. The value of integral (5) extended over the surface-elements ADB, BDC, and ADC is now given by formula (4); we have, namely,

$$\left.\begin{aligned}
\int_{ADB}(u\,dx + v\,dy + w\,dz) &= n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\omega,\\
\int_{BDC}(u\,dx + v\,dy + w\,dz) &= l\left(\frac{dw}{dy} - \frac{dv}{dz}\right)\omega,\\
\int_{ADC}(u\,dx + v\,dy + w\,dz) &= m\left(\frac{du}{dz} - \frac{dw}{dx}\right)\omega,
\end{aligned}\right\} \quad \text{........(6)}$$

where the integrations must evidently be taken in the direction $xyzx \ldots$, as indicated by the arrows in the figure. In taking the sum

$$\int_{(ADB+BDC+ADC)}(u\,dx + v\,dy + w\,dz),$$

the two integrals extended along any line, as AD, BD, or CD, common to any two of the three given surface-elements evidently cancel each other. We have, therefore

$$\int_{(ADB+BDC+ADC)}(u\,dx + v\,dy + w\,dz) = \int_{ABC}(u\,dx + v\,dy + w\,dz).$$

Replacing here the first integral by its values (6) we find

$$\int_{ABC}(u\,dx+v\,dy+w\,dz)$$

$$=\left[l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right]\omega,$$

hence

$$\sum\int_{ABC}(u\,dx+v\,dy+w\,dz)$$

$$=\sum\left\{l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}\omega,$$

where the integration is to be extended to all triangular surface-elements of the given surface S. Since the two integrations along the periphery of adjacent surface-elements must be taken in opposite directions, as an examination of the above figure will show, the expression

$$\sum\int_{ABC}(u\,dx+v\,dy+w\,dz)$$

evidently vanishes *within* the surface S, and our summation thus reduces to one round its periphery only, that is, to one round the given curve s. The above equation can, therefore, be written

$$\int_{s}(u\,dx+v\,dy+w\,dz)$$

$$=\int_{S}do\left\{l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\},$$

which is the most general form of Stokes' theorem.

At present we shall only make use of Stokes' theorem to transform the expression (20, XII.) for the number of lines of force sent through any given circuit s by any other circuit s'. The magnetic field produced by a

circuit s' of current-strength i' is now defined by formulae (24, XI.), namely,

$$\alpha_1=\frac{1}{\mathfrak{V}}\left(\frac{d\bar{q}}{dz}-\frac{d\bar{r}}{dy}\right),\quad \beta_1=\frac{1}{\mathfrak{V}}\left(\frac{d\bar{r}}{dx}-\frac{d\bar{p}}{dz}\right),\quad \gamma_1=\frac{1}{\mathfrak{V}}\left(\frac{d\bar{p}}{dy}-\frac{d\bar{q}}{dx}\right).$$

Substituting these values for α_1, β_1, γ_1 in the above expression (20, XII.) for ΣZ, we have

$$\Sigma Z=\frac{1}{\mathfrak{V}}\int M\,do\left\{\left(\frac{d\bar{q}}{dz}-\frac{d\bar{r}}{dy}\right)\cos(n,x)\right.$$
$$\left.+\left(\frac{d\bar{r}}{dx}-\frac{d\bar{p}}{dz}\right)\cos(n,y)+\left(\frac{d\bar{p}}{dy}-\frac{d\bar{q}}{dx}\right)\cos(n,z)\right\}\ldots\ldots(7)$$

If we assume that M is constant, this integral can be transformed by Stokes' theorem, and we find

$$\Sigma Z=\frac{M}{\mathfrak{V}}\int(\bar{p}\,dx+\bar{q}\,dy+\bar{r}\,dz)=\frac{M}{\mathfrak{V}}\int ds(\bar{p}\lambda+\bar{q}\mu+\bar{r}\nu),\ldots(8)$$

where ds denotes any linear-element of the given circuit s and λ, μ, ν its direction-cosines. The great advantage gained by this transformation is that ΣZ is now expressed as an integral taken round a given curve or circuit s, whereas it was formerly given by an integral (7) extended over quite an arbitrary surface S bounded by that circuit. If we denote any linear-element of the circuit s' by ds', its direction-cosines by λ', μ', ν' and the distance between that element and any linear-element ds of the given circuit s by ρ, we have then, by formulae (31, XI.),

$$p=\frac{\lambda' i'}{\sigma},\quad q=\frac{\mu' i'}{\sigma},\quad r=\frac{\nu' i'}{\sigma},$$

where σ denotes the cross-section of the circuit s', and hence by formulae (25*a*, XI.)

$$\bar{p}=i'\int\frac{\lambda' ds'}{\rho},\quad \bar{q}=i'\int\frac{\mu' ds'}{\rho},\quad \bar{r}=i'\int\frac{\nu' ds'}{\rho}.$$

Substituting these values for $\bar{p}$, $\bar{q}$, $\bar{r}$ in the above expression (8) for ΣZ, we find

$$\Sigma Z = -\frac{Mi'}{\mathfrak{V}}\iint \frac{\lambda\lambda' + \mu\mu' + \nu\nu'}{\rho}\, ds\, ds',$$

or, since $\qquad \lambda\lambda' + \mu\mu' + \nu\nu' = \cos(ds, ds'),$

$$\Sigma Z = -\frac{Mi'}{\mathfrak{V}}\iint \frac{\cos(ds, ds')\, ds\, ds'}{\rho}$$

$$= -Mi'_m \iint \frac{\cos(ds, ds')\, ds\, ds'}{\rho}, \qquad \text{.............(9)}$$

where i'_m is the current-strength measured in magnetic units.

It follows from formulae (26, XII.) and (9) that the work done by the forces that act apparently at a distance between two electric circuits is equal to the increment of the following quantity:

$$-\frac{i}{\mathfrak{V}}\Sigma Z = \frac{ii'M}{\mathfrak{V}^2}\iint \frac{\cos(ds, ds')\, ds\, ds'}{\rho}$$

$$= i_m i'_m M \iint \frac{\cos(ds, ds')\, ds\, ds'}{\rho}, \qquad \text{..........(10)}$$

(cf. also formula (18, XII.)); if its increment is positive, the displacement will be in the direction of the lines of force. This quantity (10) is, therefore, the potential of the two circuits.

We have seen on p. 233 that the action between quantities of real magnetism immersed in a fluid is inversely proportional to M, its constant of magnetic conduction, but on p. 244 that the action of electric currents on real magnetism is entirely independent of this constant. The above expression (10) shows that the action between electric currents is directly proportional to the magnetic conductivity M of the intervening medium, provided no free magnetism is excited within the given wires by the currents themselves, that is, provided the variation in the constant M can be neglected within them; this condition would not, for example, be

fulfilled in two heavy iron rings, that were traversed by electric currents, and certain parts of which were solenoidally constituted.

In the above cases M has been assumed to be constant throughout the intervening medium. If it were variable, we should then have, in addition to the forces mentioned above, those arising from the free magnetism due to the magnetic polarization of the field by the given electric currents, wherever M is either variable or discontinuous, as in non-homogeneous media or on the dividing-surfaces of adjoining media respectively (see also p. 240).

SECTION XXX. AMPÈRE'S LAW OF FORCE BETWEEN THE ELEMENTS OF ELECTRIC CURRENTS; NEUMANN'S, WEBER'S, AND VON HELMHOLTZ'S POTENTIALS.

We have seen in the preceding article that the work done by the forces, which act apparently at a distance between two circuits, during any displacement of the latter, is given by formula (10). From this let us now deduce the law of force assumed by Ampère to act between the elements of two circuits, and thus known as Ampère's law of force. This force must evidently be so chosen that the work done by or spent on the elements of either circuit during any displacement of it—let us consider here a displacement of the circuit s—gives the expression already found for the work δV_{12} actually done, namely,

$$\delta V_{12} = i_m i'_m \delta \iint \frac{\cos(ds, ds')ds\,ds'}{\rho}$$

$$= i_m i'_m \iint \delta\left(\frac{\cos(ds, ds')}{\rho}\right) ds\,ds', \quad \ldots\ldots\ldots\ldots (11)$$

where we have put $M = 1$ (air).

The given problem thus reduces to one in the theory of variation, namely, to the evaluation of the expression $\delta\left(\frac{\cos(ds, ds')}{\rho}\right)$. We have

$$\delta\left(\frac{\cos(ds, ds')}{\rho}\right) = -\frac{\cos(ds, ds')}{\rho^2}\delta\rho + \frac{1}{\rho}\delta(\cos(ds, ds')). \quad \ldots(12)$$

From the relation

$$\rho^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

it follows that

$$\delta\rho = \frac{x-x'}{\rho}\delta x + \frac{y-y'}{\rho}\delta y + \frac{z-z'}{\rho}\delta z,$$

and from the following

$$\cos(ds, ds') = \frac{dx}{ds}\cdot\frac{dx'}{ds'} + \frac{dy}{ds}\cdot\frac{dy'}{ds'} + \frac{dz}{ds}\cdot\frac{dz'}{ds'},$$

that $\delta[\cos(ds, ds')] = \frac{dx'}{ds'}\frac{d(\delta x)}{ds} + \frac{dy'}{ds'}\frac{d(\delta y)}{ds} + \frac{dz'}{ds'}\frac{d(\delta z)}{ds}$.

Substituting these values for $\delta\rho$ and $\delta(\cos(ds, ds'))$ in the above expression (12) we have

$$\delta\left(\frac{\cos(ds, ds')}{\rho}\right) = \frac{1}{\rho}\left[\frac{dx'}{ds'}\frac{d(\delta x)}{ds} + \frac{dy'}{ds'}\frac{d(\delta y)}{ds} + \frac{dz'}{ds'}\frac{d(\delta z)}{ds}\right]$$
$$-\frac{\cos(ds, ds')}{\rho^3}[(x-x')\delta x + (y-y')\delta y + (z-z')\delta z],$$

by which formula (11) can be written

$$\delta V_{12} = i_m i'_m M\left\{\iint\frac{ds\,ds'}{\rho}\left[\frac{dx'}{ds'}\frac{d(\delta x)}{ds} + \frac{dy'}{ds'}\frac{d(\delta y)}{ds} + \frac{dz'}{ds'}\frac{d(\delta z)}{ds}\right]\right.$$
$$-\iint\frac{\cos(ds, ds')\,ds\,ds'}{\rho^3}$$
$$\left.\times[(x-x')\delta x + (y-y')\delta y + (z-z')\delta z]\right\}. \quad \ldots\ldots\ldots\ldots(13)$$

The last integral of this expression (13) for δV_{12} has the desired form, namely,

$$\iint\frac{\cos(ds, ds')dsds'}{\rho^3}[(x-x')\delta x+(y-y')\delta y+(z-z')\delta z]$$

$$=\int ds\int ds'dR_1[\cos(\rho, x)\delta x+\cos(\rho, y)\delta y+\cos(\rho, z)\delta z]\text{..}(14)$$

where dR_1 is the resultant force due to the given integral, that acts between the elements ds and ds'.

To bring the first integral into the desired form (14) several transformations are necessary. Take its first term and put

$$\iint\frac{1}{\rho}\frac{dx'}{ds'}\frac{d(\delta x)}{ds}dsds'=U_1.$$

As $\dfrac{dx'}{ds'}$ is quite independent of the integration with regard to s, we can write

$$U_1=\int dx'\int\frac{1}{\rho}d(\delta x).$$

Integrating partially with regard to s, we have

$$U_1=\int dx'\left|\frac{1}{\rho}\delta x\right|-\int dx'\int d\left(\frac{1}{\rho}\right)\delta x.$$

As the given integration is to be taken round the circuit s, from any given point back to that point, the quantity $\left|\frac{1}{\rho}\delta x\right|$ assumes the same value at both upper and lower limits; the first integral thus vanishes, and we have

$$U_1=-\int dx'\int d\left(\frac{1}{\rho}\right)\delta x=\int dx'\int\frac{d\rho}{\rho^2}\delta x.\ \text{.........}(15)$$

The desired form (14) for U_1 must moreover contain the factor

$$\cos(\rho, x)=\frac{x-x'}{\rho}$$

under the integral-signs; to effect this, we form the

differential of the expression $\left(\frac{x-x'}{\rho^2}\cdot\frac{d\rho}{ds}\right)$ with regard to s', namely,

$$d\left(\frac{x-x'}{\rho^2}\,\frac{d\rho}{ds}\right)=\frac{x-x'}{\rho^2}d\left(\frac{d\rho}{ds}\right)-\left(\frac{1}{\rho^2}dx'+\frac{2(x-x')}{\rho^3}d\rho\right)\frac{d\rho}{ds}, \quad \text{..(16)}$$

integrate round the circuit s', and we find

$$\int d\left(\frac{x-x'}{\rho^2}\cdot\frac{d\rho}{ds}\right)=\int\frac{x-x'}{\rho^2}d\left(\frac{d\rho}{ds}\right)-\int\frac{1}{\rho^2}\,\frac{d\rho}{ds}dx'-2\int\frac{x-x'}{\rho^3}\,\frac{d\rho}{ds}d\rho.$$

As the first integral evidently vanishes, the integration being round a closed curve, this equation can be written

$$\int\frac{1}{\rho^2}\,\frac{d\rho}{ds}dx'=\int ds'\frac{x-x'}{\rho^3}\left(\rho\frac{d^2\rho}{ds\,ds'}-2\frac{d\rho}{ds}\,\frac{d\rho}{ds'}\right),$$

and hence the above expression (15) for U_1 as follows:

$$U_1=\iint ds\,ds'\left(\frac{x-x'}{\rho^3}\right)\left(\rho\frac{d^2\rho}{ds\,ds'}-2\frac{d\rho}{ds}\cdot\frac{d\rho}{ds'}\right)\delta x, \quad \text{......(17)}$$

which is the desired form.

To transform the expression

$$\rho\frac{d^2\rho}{ds\,ds'}-2\frac{d\rho}{ds}\cdot\frac{d\rho}{ds'},$$

we make use of the following relations:

$$\frac{d\rho}{ds}=\frac{x-x'}{\rho}\,\frac{dx}{ds}+\frac{y-y'}{\rho}\,\frac{dy}{ds}+\frac{z-z'}{\rho}\,\frac{dz}{ds}=\cos(ds,\rho),$$

$$\frac{d\rho}{ds'}=-\cos(ds',\rho),$$

and

$$\begin{aligned}\frac{d^2\rho}{ds\,ds'}&=-\frac{1}{\rho}\Big[\left(\frac{dx}{ds}\cdot\frac{dx'}{ds'}+\frac{dy}{ds}\cdot\frac{dy'}{ds'}+\frac{dz}{ds}\cdot\frac{dz'}{ds'}\right)\\&\qquad+\left(\frac{x-x'}{\rho}\,\frac{dx}{ds}+\frac{y-y'}{\rho}\,\frac{dy}{ds}+\frac{z-z'}{\rho}\,\frac{dz}{ds}\right)\Big]\\&=-\frac{1}{\rho}[\cos(ds,ds')-\cos(ds,\rho)\,.\,\cos(ds',\rho)],\end{aligned}$$

and we find

$$\rho\frac{d^2\rho}{ds\,ds'}-2\frac{d\rho}{ds}\cdot\frac{d\rho}{ds'}=3\cos(ds,\rho)\cos(ds',\rho)-\cos(ds,ds'), \quad (18)$$

by which the above integral (17) for U_1 can be written

$$U_1=\iint ds\,ds'\frac{x-x'}{\rho^3}[3\cos(ds,\rho)\cos(ds',\rho)-\cos(ds,ds')]\delta x.$$

The second and last terms of the first integral, formula (13), similarly treated give

$$U_2=\iint ds\,ds'\frac{y-y'}{\rho^3}[3\cos(ds,\rho)\cos(ds',\rho)-\cos(ds,ds')]\delta y,$$

$$U_3=\iint ds\,ds'\frac{z-z'}{\rho^3}[3\cos(ds,\rho)\cos(ds',\rho)-\cos(ds,ds')]\delta z.$$

Substituting these values in formula (13) we get

$$\delta V_{12}=i_m i'_m\iint ds\,ds'\frac{3\cos(ds,\rho)\cos(ds',\rho)-2\cos(ds,ds')}{\rho^2}$$
$$\times\left\{\frac{x-x'}{\rho}\delta x+\frac{y-y'}{\rho}\delta y+\frac{z-z'}{\rho}\delta z\right\},$$

where δx, δy, δz are entirely independent variations. It follows from this equation that the desired expression (11) for the work actually done during the given displacement can always be obtained by assuming that a force acts between the elements of the two circuits, whose components dX, dY, dZ are

$$\left.\begin{aligned}
dX&=i_m i'_m\frac{3\cos(ds,\rho)\cos(ds',\rho)-2\cos(ds,ds')}{\rho^2}\cdot\frac{x-x'}{\rho}ds\,ds'\\
dY&=i_m i'_m\frac{3\cos(ds,\rho)\cos(ds',\rho)-2\cos(ds,ds')}{\rho^2}\cdot\frac{y-y'}{\rho}ds\,ds'\\
dZ&=i_m i'_m\frac{3\cos(ds,\rho)\cos(ds',\rho)-2\cos(ds,ds')}{\rho^2}\cdot\frac{z-z'}{\rho}ds\,ds'
\end{aligned}\right\}; \quad (19)$$

the resultant force is thus

$$dR = i_m i'_m \frac{3\cos(ds, \rho)\cos(ds', \rho) - 2\cos(ds, ds')}{\rho^2} ds\, ds'; \quad (19a)$$

it evidently acts along the vector ρ. This law of force, which is assumed to act between the elements of electric currents, is known as Ampère's law. We must regard it, however, as merely a feature of our concrete representation—it is only a mechanical or analytical analogy or formula, by which the work done during the displacement of any electric circuit in a field generated by other electric currents may readily be determined; for we not only exclude all forces that act at a distance from our theory of electricity and magnetism, but we recognize the impossibility of any law of force existing between entirely isolated elements.

If the two elements of current are parallel to each other and their direction of flow at right angles to their vector ρ, that is, if

$$\angle(ds, ds') = 0 \text{ and } \angle(ds, \rho) = \angle(ds', \rho) = 90°,$$

the above expression (19a) for the resultant force assumes the simple form

$$dR = -\frac{2\, i_m i'_m ds\, ds'}{\rho^2}; \quad \ldots\ldots\ldots\ldots\ldots (19b)$$

If the two given elements lie in the same straight line, then

$$\angle(ds, ds') = \angle(ds, \rho) = \angle(ds', \rho) = 0,$$

and hence $$dR = \frac{i_m i'_m ds\, ds'}{\rho^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19c)$$

Ampère's law of force is not the only one that gives the desired expression (11) for the work δV_{12} done during any displacement; this becomes evident upon a more careful examination of the above development. Take, namely, the expression for the x-component of the force exerted by the circuit s' on any element ds of the circuit s, namely,

$$\sum_{s'} dX = i_m i'_m ds \int ds' \frac{3\cos(ds,\rho)\cos(ds',\rho) - 2\cos(ds,ds')}{\rho^2}\,\frac{x-x'}{\rho}, \quad (20)$$

and add to the expression under the integral sign any quantity, which, integrated round the former circuit vanishes; the resulting expression integrated round the circuit s would also give the correct value for δV_{12}. Such a quantity is

$$C\frac{d}{ds'}\left(\frac{x-x'}{\rho}\,\frac{d\rho}{ds}\right)ds'$$

$$= Cds'\left\{\frac{x-x'}{\rho^2}\,\frac{d^2\rho}{ds\,ds'} - \left(\frac{1}{\rho^2}\,\frac{dx'}{ds'} + \frac{2(x-x')}{\rho^3}\,\frac{d\rho}{ds'}\right)\frac{d\rho}{ds'}\right\}, \quad \ldots(21)$$

where C is an arbitrary constant.

By the relations on p. 279 this expression could be written

$$C\frac{d}{ds'}\left(\frac{x-x'}{\rho^2}\,\frac{d\rho}{ds}\right)ds$$

$$= Cds'\left\{\frac{x-x'}{\rho^3}[3\cos(ds,\rho)\cos(ds',\rho) - \cos(ds,ds')] - \frac{1}{\rho^2}\cos(ds',x)\cos(ds,\rho)\right\};$$

integrated round the circuit s', it would give

$$0 = \int Cds'\left\{\frac{x-x'}{\rho^3}[3\cos(ds,\rho)\cos(ds',\rho) - \cos(ds,ds')] - \frac{1}{\rho^2}\cos(ds',x)\cos(ds,\rho)\right\}.$$

Introducing this expression in formula (20), we should get

$$\sum_{s'} dX = i_m i'_m ds \int ds' \times \left\{\frac{3(C+1)\cos(ds,\rho)\cos(ds',\rho) - (C+2)\cos(ds,ds')}{\rho^2} \times \frac{x-x'}{\rho} - \frac{C\cos(ds,\rho)}{\rho^2}\cos(ds',x)\right\}.$$

It follows from this expression that the component along the x-axis of the force acting between the elements of two circuits would be

$$i_m i'_m ds ds' \left\{ \frac{3(C+1)\cos(ds,\rho)\cos(ds',\rho)-(C+2)\cos(ds,ds')}{\rho^2} \times \frac{x-x'}{\rho} - \frac{C\cos(ds,\rho)}{\rho^2}\cos(ds',x) \right\};$$

similar expressions would hold for the other two component-forces; the resultant force would thus be

$$i_m i'_m ds ds' \left\{ \frac{3(C+1)\cos(ds,\rho)\cos(ds',\rho)-(C+2)\cos(ds,ds')}{\rho^2} - \frac{C\cos(ds,\rho)}{\rho^2} \right\}.$$

This force would not, however, act along the vector ρ; every element of current would thus acquire a certain moment, that would tend to set it in rotation; this rotatory motion would increase indefinitely, and the equation of kinetic energy would thus cease to hold. This apparent paradox must now be attributed, on the one hand, to Ampère's assumption that the elements of electric currents act on one another, and, on the other hand, to the fact that certain expressions, as (21), which vanish when integrated round closed circuits, as s', have been added at will to those that have already been assumed by Ampère to hold for every element. If it were, indeed, possible to treat the elements of a current separately, it is evident that Ampère's law of force would be the one that must be accepted, since the elements of current acquire here no moment. This whole subject is, however, of interest only so far as it concerns our concrete representation, since the phenomena that are attributed by the old theory of electricity and magnetism to the action of forces at a distance are assigned by us to forces residing in the medium or ether.

We could also treat the expression (10) for the potential V_{12} similarly to the above integral (20), that is, we could add to the expression under the integral-signs any quantity, which, integrated round the circuit, vanishes; such a quantity is

$$C\frac{d^2\rho}{ds\,ds'}ds=\frac{C\cos(ds,\,\rho)\cos(ds',\,\rho)-C\cos(ds,\,ds')}{\rho}ds',$$

where C is an arbitrary constant; this added to the given expression would give

$$V_{12}=i_m i'_m\iint\frac{C\cos(ds,\rho)\cos(ds',\rho)+(1-C)\cos(ds,ds')}{\rho}ds\,ds'$$

or, if for symmetry we put

$$C=\frac{1-k}{2},$$

$$V_{12}=\tfrac{1}{2}i_m i'_m$$
$$\times\iint\frac{(1-k)\cos(ds,\rho)\cos(ds',\rho)+(1+k)\cos(ds,ds')}{\rho}ds\,ds';\;(22)$$

this expression for V_{12} was first introduced by von Helmholtz,* and is thus known as von Helmholtz's potential.

For $k=1$, V_{12} assumes the form

$$V_{12}=i_m i'm\iint\frac{\cos(ds,\,ds')ds\,ds'}{\rho},\;\ldots\ldots\ldots\ldots(23)$$

which is known as F. E. Neumann's† potential.

On putting $k=-1$ in the above expression (22), we obtain Weber's potential, namely,

$$V_{12}=i_m i'_m\iint\frac{\cos(ds,\,\rho)\cos(ds',\,\rho)ds\,ds'}{\rho}\ldots\ldots(24)$$

The potential (10), from which we started above, obtained from the general expression (22) by putting $k=0$, could be designated as Maxwell's† potential.

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* Cf. *Wissenschaftliche Abhandlungen*, v. 1, pp. 539, 567.

† Cf. same, p. 540.

CHAPTER XIV.

SECTION XXXI. MECHANISMS FOR ILLUSTRATING ELECTRO-MAGNETIC PHENOMENA; THEIR MECHANICS: CYCLES, AND LAGRANGE'S EQUATIONS OF MOTION FOR CYCLES. MONOCYCLES; A MONOCYCLIC MECHANISM.

In the derivation of the above well-known laws of electrodynamics we have followed essentially Hertz's development. Although this is undoubtedly correct, it fails to give us a clear insight into the real nature of the phenomena themselves. An exhaustive treatment of the theory of ponderable forces would naturally have to be deferred to the chapters on the electrodynamics of moving bodies and electro- and magneto-striction (see § 43). We shall, nevertheless, endeavour to show here how deep an insight can be obtained of the real nature of electro-magnetic phenomena from mechanical models. Such models or mechanisms must, however, be considered merely as means for facilitating our conceptive power and not as anything really existing; they are, indeed, to be regarded only as another feature of our concrete representation. As the mechanics of these mechanisms require a certain knowledge of the theory of cycles and their equations of motion, we shall begin with this important subject.

If a motion is imparted to a given system of bodies, not only the bodies themselves will change their position in space, but their state will, in general, also be altered.

This will not, however, always be the case, or at least it is not necessary that it should be so; an absolutely steady current can, for example, flow for days through a wire, and the position, temperature, flow of heat, magnetic state in surrounding masses of iron, in fact every state or condition perceptible to either our senses or our best physical instruments will remain unchanged. The motion to which we attribute this phenomenon must, therefore, be a steady motion, that is, such that as soon as any particle leaves its place another similar particle advancing with the same velocity in the same direction immediately succeeds it, so that in spite of the continuous motion no change becomes perceptible at any point. von Helmholtz calls such a motion a cyclic motion and a system, within which all motions are cyclic, a cyclic system or cycle. Examples of cyclic systems are: a rigid body constructed symmetrically with regard to a given axis and rotating with constant velocity about it; several such bodies coupled together by driving belts; a fluid flowing with uniform velocity in a closed canal, etc.

Before proceeding to the treatment of the mechanics of cycles, let us first consider briefly the equations of motion for arbitrary mechanical systems. If the position and state of any system of bodies are determined by n independent variables $l_1, l_2, \ldots, l_n$, the system is said to possess n degrees of freedom, and the l's are known as its n generalized coordinates. Thus a material particle possesses one degree of freedom when constrained to move along a curve, and three when free to move in space, whereas a rigid body possesses six degrees of freedom.

If L denotes the force (generalized) that tends to increase any coordinate l, the total work δA done during any variation δl in l will evidently be

$$\delta A = \Sigma L \delta l;$$

this is equal to the increment δT of the kinetic energy of

the system. In general, the particles of any system can always be conceived as connected in such a manner with n so-called driving-points constrained to move on prescribed curves that the displacement of any given driving-point will produce a change δl in only one variable, and that δl will also be equal to the distance traversed by that driving-point. For a material particle free to move in space, its projections on the coordinate-axes could be chosen as its three driving-points; for a rigid body rotating on a fixed axis, one of the driving-points could be taken at unit distance from that axis. If our limited knowledge of the mechanism of the given system should not permit of the above definition of L, namely as the force that tends to increase the coordinate l—suppose that the given system contain a galvanic element and that l denote the total quantity of electricity that had flowed through it, we should then have to define L as the quotient $\delta A/\delta l$, where δA denotes the work done during the increment δl; and such a definition could involve no doubt whatever, since the work done is something that can always be uniquely defined.

If we take it for granted that the given system is a mechanical system obeying the general equations of analytic mechanics, we can, in spite of our ignorance of its real mechanism, apply Lagrange's general equations of motion to it, namely,

$$L=\frac{d}{dt}\frac{\partial T}{\partial l'}-\frac{\partial T}{\partial l}, \quad \text{.....................(1)}$$

where the kinetic energy T is to be regarded as a function of the n generalized coordinates and their differential-quotients with regard to the time, $l'_1, l'_2, \ldots, l'_n$. The quotient $\delta T/\delta l'$ is usually known as the moment of the given system with regard to the coordinate l' and denoted by λ. For the proof of equations (1), refer to: Lagrange, *Mècanique Analytique*, part II., § 4; Thomson and Tait's *Natural Philosophy*, v. I., part II, new edition § 318 (24); Maxwell's *Treatise on Electricity and Mag-*

netism, v. II., § 571 (new edition); and Jacobi, *Vorlesungen über Dynamik* (eighth lecture).

To determine the modified form of Lagrange's equations of motion for the special case, where the given system is a cyclic one, we observe that every cyclic motion is characterized by the fact that as soon as any particle leaves its place another similar particle, advancing with the same velocity in the same direction, immediately succeeds it and that the state of the system thus remains unchanged, and hence that its kinetic energy T cannot be a function of l; on the other hand, we shall designate l as a cyclic coordinate whenever it represents such a cyclic motion, that is, when the state of the system, and hence its kinetic energy, undergo no change from its variation. T will, however, generally contain l', since the kinetic energy of any system increases as its cyclic motion becomes more rapid.

The state of any cycle can, in general, be determined by a given number of cyclic coordinates. If the given state is characterized by cyclic coordinates only, it can never attain to any other state, since no change of state can be brought about by the cyclic coordinates themselves. It is possible, however, that, although the state of a given system may not be absolutely steady and cannot thus be exactly determined by cyclic coordinates, the changes in its state take place so slowly that the motion of the system during any given period of short duration differs only infinitesimally from a steady or cyclic motion. To define such a state Helmholtz introduces slowly changing coordinates or parameters k—they correspond to the variable parameters in the theory of the variation of the constants; these parameters are supposed to change so slowly that their differential quotients with regard to the time can be neglected. It thus follows that the kinetic energy T of such a system will be a function of the k's, but not of their derivatives with regard to the time, the k''s. We have, therefore,

$$T=f(k, l') \quad \text{.........................(2)}$$

for the given system. From the slow variation of the k's it follows that they can be regarded as constant during a considerable lapse of time and thus the motion itself as approximately cyclic (during that period). After a greater lapse of time the parameters k will assume other values, and the motion can again be regarded as cyclic. We can, therefore, define a cycle as a system containing only cyclic and slowly changing coordinates or parameters. By equation (2) our general equations of motion (1) for cycles reduce to

$$L=\frac{d}{dt}\frac{\partial T}{\partial l'}, \quad K=-\frac{\partial T}{\partial k}, \quad \text{.................(3)}$$

where L and K denote the forces that tend to increase the cyclic coordinates l and the parameters k respectively.

The simplest case of cyclic motion is where we have a single cyclic coordinate l; such a system is called a monocycle. The positions of all parts of such a system, except those due to the slow variation of the parameters k, will then be determined by the position of a single driving-point defined by that coordinate. If the driving-point is at rest, all parts of the system will remain at rest, whereas, if it is moving with a given velocity l', the motions of all its masses will be determined by that velocity. The value of the several parameters is, of course, always supposed to be not only given but approximately constant for the given period. Such a monocycle can consist of any number of particles, m_1, $m_2, \ldots m_p$ connected arbitrarily with the given driving-point and with one another. We have seen above that the kinetic energy T of a cycle is a function of the velocity l' and the k's only; it is thus evident that, if every particle retains the same state of motion during its entire motion, its velocity v will also be a function of l' and the k's only; whereas, if it assumes different velocities along its path, we can maintain only that the velocity at any given point of space will be a function of these variables; this follows from the expression for T, which is a

sum extended to all particles of the system; take a fluid flowing cyclically in a closed canal of variable cross-section as an illustration of such a monocycle.

In mechanical systems given conditions often exist between the parameters. By means of these conditional equations certain parameters can be eliminated, as many, in fact, as there are such conditions. We shall not, however, perform the eliminations here, but shall retain the given parameters and the forces acting on them, even when the latter vanish, in our equations, since then cases can hardly be realizable, where the velocities of the given particles are other than linear functions of the cyclic velocity l'; this will also hold for bicycles and polycycles, as we shall observe later. The assumption of any other than a linear relation between the velocity v and the cyclic coordinate l' would at first sight seem to lead to mechanical absurdities; if, however, other motions were ever discovered, we should only have to conclude that the motions which give rise to magnetic and electric phenomena do not belong to this purely hypothetical category. We can put then

$$v_i = a_i l' \quad \text{..........................(4)}$$

and
$$T = \frac{l'^2}{2}\Sigma m_i a_i^2 = \frac{A}{2}l'^2, \quad (i=1, 2, \dots p), \quad \text{.........(5)}$$

where the coefficients a are to be regarded as functions of the parameters k. Equations (3) thus reduce to

$$L = \frac{d(Al')}{dt}, \quad K_h = -\frac{l'^2}{2}\frac{\partial A}{\partial k_h} \cdot (h=1, 2, \dots n). \quad \text{.....(6)}$$

To illustrate the meaning of the above principles, let us examine the monocyclic mechanism represented in the annexed figure: a cylindrical shaft turned on its vertical axis by a handle carries a horizontal spoke, along which a mass m of very small volume can slide without friction. A string attached to the mass m passes over a small pulley at the junction of the shaft and spoke to a

scale-pan S carrying a weight p. We assume that the shaft, spoke and string are without mass. Let r, the distance of the mass m from the shaft, and l, the angle

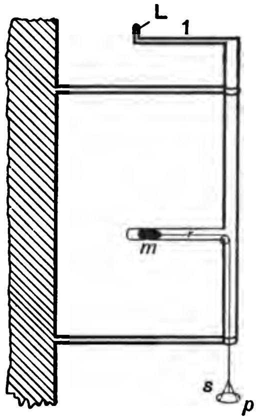

FIG. 34.

through which the shaft has been turned from a given initial position, be chosen as our generalized coordinates.

We have then for the kinetic energy T of the system

$$T = \frac{m}{2}(r'^2 + r^2 l'^2).$$

Lagrange's equations of motion thus give

$$R = mr'' - mrl'^2 \quad \text{and} \quad L = mr^2 l'' - 2mr'l'.$$

If the distance of the handle from the shaft is chosen as unity, the force (moment) L, which tends to increase the angle l, can be supposed to act on the handle in a tangential direction. The centrifugal force R tends to increase the distance r.

If initially $L = 0$ and p exactly balances the centrifugal force R, then

$$l'' = r' = 0, \text{ and hence } p = mrl'^2.$$

Next, let so little force be brought to act on the handle of the shaft that l'' still remains very small in comparison to l'; r' will then be very small in comparison to rl', and hence r'' in comparison to rl'^2. p and T thus assume here the approximate values

$$p = -R = mrl'^2, \quad T = \frac{m}{2}r^2l'^2.$$

The work dQ imparted to the system by the force L during the interval dt is spent here partly in increasing its kinetic energy $mr^2l'^2/2$ and partly in raising the weight p through the distance dr. We have, therefore,

$$dQ = \frac{m}{2}d(r^2l'^2) + pdr = mr^2l'dl' + 2ml'^2rdr,$$

hence $$\frac{dQ}{T} = d\log(r^4l'^2), \quad\quad (7)$$

which is a complete differential.

The given mechanism is a monocycle, since its cyclic velocity l' is very large, its cyclic acceleration l'' and parametric velocity r' very small, and hence its kinetic energy T a function of l' and r only.

It is possible to perform four processes analogous to Carnot's cycle with the above monocyclic mechanism; in the two corresponding to an isothermal process the angular velocity and the distance of the mass m from the shaft are varied in such a manner that the kinetic energy of rotation remains constant, whereas in the two adiabatic processes no work is performed on the shaft and the angular momentum mr^2l' is thus kept constant.

J. J. Thomson has divided the coordinates of a mechanical system into two classes, controllable and unconstrainable coordinates. The former are any coordinates, which can be acted on directly from without, as those belonging to the volume of a body, to the charge of electricity on its surface, etc., whereas the latter define

the internal or circulating motions within it, as those which fix the position of its molecules and thus define its (thermal) state; the latter are, therefore, often called molecular coordinates. In the above the l' is the unconstrainable or molecular coordinate and the k's (r's) the controllable. As in the mechanism just considered, the motions defined by the controllable coordinates are supposed to be very slow in comparison to the molecular motions or those given by the unconstrainable coordinates; thus any changes in the volume of a body upon being heated are conceived to be very small compared to its molecular motions called heat. Analogously, we shall suppose in the following that the visible motions of charged conductors and magnets are very slow in comparison to those internal or circulating motions known as electricity (magnetism).

According to Maxwell an electric current is a monocycle, its flow being given by a cyclic coordinate l and its current-strength by the derivative of that coordinate with regard to the time, l'. The force L that acts on this cyclic coordinate and thus gives rise to the cyclic motion is called the electromotive force. The given motion is not confined to the wire only, but pervades the surrounding ether; the motion in the latter may be changed by varying the position or configuration of the wire carrying the current or the relative position of the surrounding bodies, as masses of iron; these changes are, of course, determined by the slowly changing coordinates or parameters k. It follows from the considerations on pp. 289 and 290 that the velocity of the particles that pass through any given point of the wire or the surrounding ether are proportional to l'; whereas the factors a must be functions of the parameters k, that is, of the configuration of the wire and the position of the surrounding bodies. The K's are the forces that must act on the system from without and keep the values of l' and the given parameters either constant or so nearly so that the k's and l'' assume only such values

as are consistent with our above equations. The $-K$'s are the forces to which the cyclic motion gives rise—the centrifugal force of the mass m of the monocycle just described is such a force; they are held in equilibrium by the external forces, the K's. Similarly, L is the force that must act from without on the current (the handle of the given mechanism).

Up to this point we have assumed that our system is free from all retardations to motion or resistances (friction). To complete our analogy between the monocycle and the electric current we must, therefore, introduce such retardations. Retardations to motions due to changes of the k's, such as the friction between parts of the circuit and neighbouring masses of iron, could be included under the K's; direct retardations to the cyclic motion itself must, however, be explicitly introduced into our equations. If we denote the sum of all the retardations to the given cyclic motion by W, the given force by L_m, and that which would produce the given motion, provided all direct retardations were wanting, by L_0—this force is determined by equations (3)—we can then write

$$L_m = L_0 + W = \frac{d}{dt}\frac{\partial T}{\partial l'} + W$$

or briefly

$$L = \frac{d}{dt}\frac{\partial T}{\partial l'} + W. \qquad (8)$$

If the current-strength, configuration of the wire and position of the surrounding bodies remain constant, that is, if l' and $\frac{\partial T}{\partial l'}$ remain constant, L will be equal to W and the electromotive force will be determined by Ohm's law.

SECTION XXXII. MECHANICS OF BICYCLES. A BICYCLIC MECHANISM; ITS ANALOGY TO ELECTRIC CURRENTS. A BICYCLIC MECHANISM FOR WHICH THE COEFFICIENTS A, B, C ARE INDEPENDENT OF ONE ANOTHER.

Let us next consider a bicyclic system or bicycle, that is, a system defined by two cyclic coordinates. The position of every particle of such a system is now determined by an arbitrary number n of parameters k and by two cyclic coordinates l_1 and l_2. We shall assume here as in the preceding article that the velocity v_i of any particle m_i is a linear function of the cyclic velocities l_1' and l_2', namely, that

$$v_i = a_i l_1' + b_i l_2', \quad i = 1, 2, \ldots n, \ldots\ldots\ldots\ldots (9)$$

where a_i and b_i are functions of the parameters k.

Putting

$$A = \sum_{i=1}^{i=n} m_i a_i^2, \quad B = \sum_{i=1}^{i=n} m_i b_i^2, \quad C = \sum_{i=1}^{i=n} m_i a_i b_i, \ldots (10)$$

we have then $\quad T = \frac{A}{2} l_1'^2 + \frac{B}{2} l_2'^2 + C l_1' l_2', \ldots\ldots\ldots\ldots\ldots\ldots (11)$

and thus the following values for the momenta λ_1 and λ_2:

$$\lambda_1 = \frac{\partial T}{\partial l_1'} = A l_1' + C l_2', \quad \lambda_2 = \frac{\partial T}{\partial l_2'} = C l_1' + B l_2'.$$

By Lagrange's equations of motion (1) we find then the following expressions for the forces:

$$\left.\begin{aligned} L_1 &= \frac{d\lambda_1}{dt} + W_1 = \frac{d}{dt}(A l_1' + C l_2') + W_1 \\ L_2 &= \frac{d\lambda_2}{dt} + W_2 = \frac{d}{dt}(C l_1' + B l_2') + W_2 \\ K &= -\frac{\partial T}{\partial k} = -\frac{l_1'^2}{2}\frac{\partial A}{\partial k} - \frac{l_2'^2}{2}\frac{\partial B}{\partial k} - l_1' l_2' \frac{\partial C}{\partial k} \end{aligned}\right\} \ldots\ldots (12)$$

L_1 and L_2 are the forces that tend to increase the cyclic coordinates l_1 and l_2 respectively; K is the force acting on the parameter k. W_1 and W_2 are the total retardations arising from any changes in the first and second cyclic coordinates respectively. If we assume that the mechanisms, which transfer the motions from the two driving-points to the several masses of the system, encounter no resistance, but that only the driving-points are retarded in their motion, W_1 will then be a function of the motion of the first driving-point and W_2 a function of that of the second, that is, the motion of the one driving-point will have no effect whatever on the value of the other W and vice versa.

The above equations (12) are quite general. However the material particles, whose motion appeals to us as an electric current, may be constituted, and however complicated the connection of these particles with one another and the character of their motions, as long as the motions themselves are cyclic and obey the fundamental principles of mechanics, we know that they must always satisfy the given equations (12), for the latter only formulate, as it were, the principles postulated.

To demonstrate the meaning and significance of equations (12) and to illustrate their application, let us conceive the following ideal mechanism: three tubes are arranged coaxially one above the other; the ends of the middle tube are fitted into the upper and lower ones and the latter are supported by a frame in such a manner that they can be turned on their mutual axis by handles similar to the one attached to the mechanism represented in figure 34. Each tube carries a horizontal spoke, along which a mass m can slide without friction; a string fastened to this mass passes over a small pulley close to the spoke, and a weight p is attached to its free end. We denote the masses carried by the upper and lower tubes by m_1 and m_2 respectively and that by the middle one by m_3, and the distances of these masses from the axis of the tubes by r_1, r_2, and r_3 respectively. Both

the upper and lower tubes carry also a horizontal circular disc, and the middle one a second horizontal spoke, on whose axis a vertical circular disc can rotate; the radius of this disc is such that its edge rests lightly against the horizontal discs, upon which the former when rotating is supposed to roll without slipping and without friction. Compare the annexed figure.

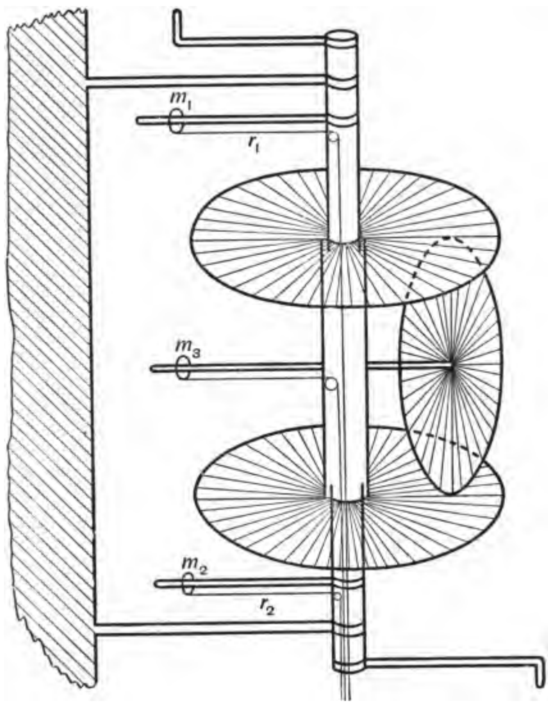

FIG. 35.

Let us examine the motion imparted to the middle tube of this mechanism upon setting its upper and lower handles in rotation. By turning both handles through

equal angles in the same direction, the middle tube is revolved through the same angle in the same direction, but the vertical disc undergoes no rotation whatever. If we turn both handles through equal but opposite angles, the middle tube remains at rest and the vertical disc is revolved. To find the motion imparted to the middle tube by any angular rotations, l_1 and l_2, of the upper and lower handles taken in the same direction, we first turn both handles in the same direction through the angle $(l_1+l_2)/2$, then the upper one through the additional angle $(l_1-l_2)/2$, and lastly, the lower one back through same angle, $(l_1-l_2)/2$; as the last two rotations counteract each other, it is evident that the middle tube is hereby turned through the angle $(l_1+l_2)/2$. The angular rotation and hence the velocity of the middle tube are thus always equal to the arithmetical mean of those of the other two.

If we assume that the above mechanism, with the exception of the three masses m_1, m_2, and m_3, is without mass, its state will be uniquely determined by the two cyclic coordinates l_1 and l_2 and the three parameters r_1, r_2, and r_3. The velocities of these masses are evidently

$$v_1=r_1l_1', \quad v_2=r_2l_2', \quad v_3=r_3\frac{l_1'+l_2'}{2}.$$

Formulae (9) and (10) thus give

$$a_1=r_1, \quad a_2=b_1=0, \quad b_2=r_2, \quad a_3=b_3=r_3/2$$

and hence

$$A=m_1r_1^2+\frac{m_3r_3^2}{4}, \quad B=m_2r_2^2+\frac{m_3r_3^2}{4}, \quad C=\frac{m_3r_3^2}{4}. \quad \text{..(13)}$$

For $m_1=m_2=\frac{m_3}{4}=m$ these coefficients assume the simple form

$$A=m(r_1^2+r_3^2), \quad B=m(r_2^2+r_3^2), \quad C=mr_3^2.$$

Substituting the values (13) for A, B, C in our general formulae (11) and (12) we have

$$\left.\begin{aligned}
T&=\left(\frac{m_1r_1^2}{2}+\frac{m_3r_3^2}{8}\right)l_1'^2+\left(\frac{m_2r_2^2}{2}+\frac{m_3r_3^2}{8}\right)l_2'^2+\frac{m_3r_3^2}{4}l_1'l_2'\\
L_1&=\frac{d}{dt}\left[\left(m_1r_1^2+\frac{m_3r_3^2}{4}\right)l_1'+\frac{m_3r_3^2}{4}l_2'\right]+W_1\\
L_2&=\frac{d}{dt}\left[\frac{m_3r_3^2}{4}l_1'+\left(m_2r_2^2+\frac{m_3r_3^2}{4}\right)l_2'\right]+W_2\\
R_1&=-m_1r_1l_1'^2,\quad R_2=-m_2r_2l_2'^2,\quad R_3=-\frac{m_3r_3}{4}(l_1'+l_2')^2
\end{aligned}\right\};(14)$$

W_1 and W_2 denote here the frictions in the sockets, where the ends of the middle tube are fitted into those of the other two.

To examine the behaviour of the given mechanism first set the upper tube in rotation by means of its handle and seek the force L_2 that must be brought to act on the lower handle, that it may remain at rest, that is, that $l_2'=0$. Here W_2 must evidently vanish and equations (12) and (14) thus give

$$L_2=\frac{d}{dt}(Cl_1')=\frac{d}{dt}\left(\frac{m_3r_3^2}{4}l_1'\right);$$

from this expression for L_2 it follows that, as long as the r's and the velocity of the upper handle remain constant, no force whatever will be required to keep the lower tube at rest, whereas, if the velocity of the upper handle increases, a force must be brought to act on the lower one, whose direction is that of the force L_1 and whose strength is directly proportional to $\frac{dl_1'}{dt}$ and to the coefficient C. If this force were wanting, the lower tube would evidently be turned in an opposite direction to that of the upper handle during any increment in the velocity of the latter; similarly, if the velocity of the

upper handle decreases, the lower tube is turned in the direction in which the former is revolving. The same effects can also be produced by keeping the angular velocity of the upper handle constant and by increasing or decreasing the value of the coefficient C; the latter can be effected by a proper manipulation of the string attached to the mass m_3; namely, by diminishing the tension of the string, that is, by increasing r_3 and hence C, we set the lower tube rotating in the opposite direction to that of the upper handle, and by decreasing r_3 and hence C in the same direction.

The above analogy extends even further. By equations (12) the forces K that act on the parameters k are proportional to $l_1'^2$ for the parameters contained in A, to $l_2'^2$ for those contained in B, and to $l_1'l_2'$ for those in C. Take the above mechanism: if r_1 changes and r_2 and r_3 remain constant, a force R_1 proportional to $l_1'^2$ and tending to increase r_1 will appear; a force $-R_1$ must thus be brought to act on this parameter (r_1) from without, if the equilibrium of the system is to be maintained. On the other hand, if r_3 and hence the coefficient C increase, an external force $-R_3$ proportional to $l_1'l_2'$ and tending to decrease r_3 must be introduced. The analogous is true of two electric currents: a change in the self-induction of the primary or secondary circuit due to a change in any coordinate (parameter) gives rise to a ponderable force proportional to the square of the current-strength of the given circuit, whereas a change in the coefficient of mutual induction arising from any varying parameter calls forth a ponderable force proportional to the product of their current-strengths.

A more careful comparison between induced currents and the given mechanism shows that we cannot assume that the retardations W_1 and W_2 to the forces L_1 and L_2 are represented by the frictions in the sockets between the tubes, but that we must introduce resistances that are proportional to the velocities l_1' and l_2';

such resistances could in fact be approximately realized by attaching windsails to the handles of the upper and lower tubes.

The student is not to infer from the above analogy between the behaviours of electric currents and the given mechanism that the former are to be conceived to be similar to the latter or in fact to any mechanism constructed of rigid rods, flexible strings, fluids, attracting or repulsive centres, etc. On the other hand we cannot presume to know anything about the construction or the material for the construction of the electric mechanism; this is indeed the very reason why we have constantly been obliged to introduce so many mechanisms or dynamical representations, features of our so-called concrete representation, to illustrate the manifold phenomena exhibited by such a complex mechanism as the electric current.

By varying the parameters r_1 or r_2 of the above mechanism we change only the coefficients A or B respectively. By varying the parameter r_3 however we change not only the coefficient C but also the coefficients A and B (cf. formulae (13)). It is now desirable to make these coefficients entirely independent of one another, that is, to choose our three driving-points in such a manner that the motion of the first produces a change in A and hence in r_1 only, that of the second one in B and hence in r_2 only, and that of the third one in C only; to effect this it is evident that the motion of the third driving-point must be such that the expressions

$$m_1r_1^2+m_3r_3^2/4 \quad \text{and} \quad m_2r_2^2+m_3r_3^2/4$$

always remain constant. If for simplicity we assume that

$$m_1=m_2=\frac{m_3}{4},$$

these conditions reduce to:

$$r_1^2+r_3^2=c_1, \quad r_2^2+r_3^2=c_2.$$

To effect this in our mechanism we cut an inverted T- (⊥) shaped slit in a sheet of copper. A pin c, to which the ends of two narrow strips of copper are pivoted, slides in the vertical branch of this slit. Another pin passes through a narrow slit cut in each strip and through the horizontal branch of the slit in the copper sheet; these pins a and b slide not only in the slits of the strips but in that of the copper sheet. The free ends of the strings attached to the masses m_1 and m_2 are secured to the pins a and b in the manner indicated in the annexed figure:

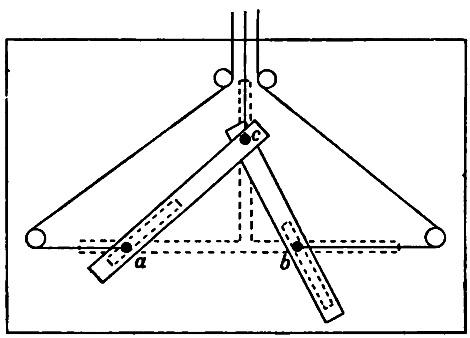

FIG. 36.

they pass namely from the given pins over small pulleys pivoted to the copper sheet near the ends of the horizontal slit, then in between two pulleys just above the vertical slit, and lastly directly upwards to the masses m_1 and m_2. The free end of the string attached to the mass m_3 is secured to the pin or driving-point c; let its vertical distance over the horizontal slit be denoted by z.

If the strings attached to the three masses are of such lengths that each mass is pulled in on to the central axis, when its respective pin is placed at the junction of the horizontal and vertical slits in the copper sheet, the conditions

$$r_1^2+r_3^2=\overline{ac}^2 \quad \text{and} \quad r_2^2+r_3^2=\overline{bc}^2$$

will then always be satisfied. To bring about a change in the lengths $\overline{ac}$ and $\overline{bc}$ without displacing the driving-point c and conversely a displacement of this driving-point without varying these lengths, imagine two sheets of copper parallel to the given sheet, the one at the variable distance x in front of the latter and the other at the variable distance y behind it, two very small tubes soldered to the pin c, the one parallel to the slit of the copper strip, but somewhat shorter than it, and the other erected at right angles to the copper sheets and extending almost to that of the latter, whose position is determined by the variable x, a third tube soldered to the pin a and fitted into the former of the given tubes in such a manner that it can slide in and out—let the pin b be similarly connected with the other copper sheet—and, lastly, a fourth tube, whose one end is similarly fitted into the latter of the first two tubes, and whose other is secured to the copper sheet determined by the variable y in such a manner that it is always kept at right angles to this sheet but that it can be displaced within it; moreover, imagine that a pliable non-elastic string or wire passes through all four tubes, and that its one end is secured to the pin a and its other to the given copper sheet at its junction with the fourth tube.

The state of the above mechanism is evidently defined by the three independent variables x, y, z, whereby a change in any one of these variables evidently produces a change in only one of the coefficients A, B, C. By decreasing x or y we decrease the parameter r_1 or r_2 respectively, and hence only the coefficient A or B respectively, whereas by displacing the pin c we change only the coefficient C. If the three tubes of the given mechanism are now set rotating, the force, which is proportional to $l_1'^2$, will thus act on only the variable or driving-point x, that proportional to $l_2'^2$ on y only, and that proportional to $l_1'l_2'$ on z only, whereas no force will act on z when either the upper or lower tube is at rest; x, y, z are therefore the three driving-points sought.

SECTION XXXIII. PRACTICAL CONSTRUCTION OF BICYCLIC MECHANISMS; THOSE OF MAXWELL, LORD RAYLEIGH AND BOLTZMANN. EXPLANATION OF MAGNETIC PHENOMENA UNDER THE ASSUMPTION OF THE NON-EXISTENCE OF REAL MAGNETISM.

The first apparatus for illustrating the effects of induced currents was constructed by Maxwell and is now preserved in the Cavendish laboratory, Cambridge; it is roughly sketched in the annexed figure. The bevel-

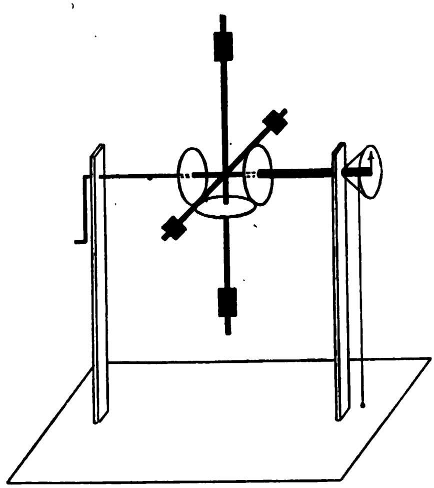

FIG. 37.

wheel on the left is soldered to the horizontal shaft terminating in the handle; the shaft itself runs between the two supports and is turned by this handle. The four rods are soldered to a tube, fitted round the first shaft and capable of rotation upon it. These rods carry weights, which can be clamped to them at any distance from the shaft, so that the inertia of the system can be

changed at will. The second bevel-wheel, that on the right, is soldered to a second tube, which rotates on the first as axis. The third bevel-wheel is attached to one of the four rods in such a manner that it can rotate freely upon it, its teeth playing directly into those of the other two wheels—the rod passes through the centre of this bevel-wheel, as indicated in the figure. The resistance to the rotation of the second bevel-wheel can be effected by winding an elastic band round its axis and securing its free end to the stand. Lastly, the pointer attached to the second tube records on a fixed dial or disc the rotation imparted to the second bevel-wheel. The rotation of the bevel-wheel driven by the handle represents the primary circuit and that of the second bevel-wheel the secondary circuit. It is evident that in starting up the apparatus the second wheel is rotated in a direction opposite to that of the first and in stopping it in the same direction, whereas as long as the velocity of rotation of the first bevel-wheel is kept constant, the second wheel remains at rest.

A still simpler apparatus than the above has been described by Lord Rayleigh in the Nov. 1890 number of the "Physical Society of London." It consists of three parallel bars, each carrying a mass or carriage, and of one

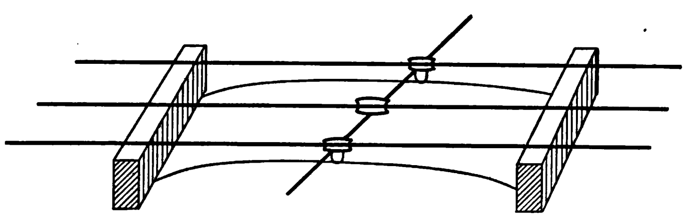

FIG. 38.

cross-bar, which passes through swivels soldered to the carriages, so that the latter are always kept in a straight line, as roughly indicated in the annexed figure. To obtain the best results the carriage on the middle parallel

bar should be very massive in comparison to the other two; this carriage corresponds to the four weighted rods of Maxwell's apparatus. The smaller carriages on the other two bars represent the primary and secondary circuits; an accelerating motion of the one will give rise to a motion of the other in the opposite direction, etc. More effective resistances than the frictions between the carriages and their respective bars can be obtained by introducing stretched cords or wires parallel to and directly beneath them, as indicated in the figure.

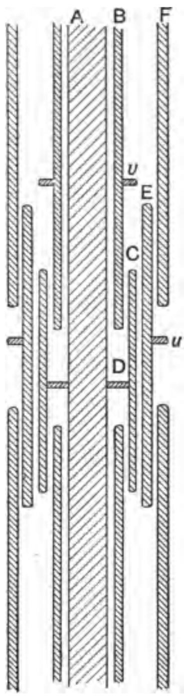

FIG. 39.

The chief shortcoming of the apparatuses just described is that the parameters cannot be changed. This has been accomplished in a model recently constructed by Boltzmann. It consists of three similar monocycles coupled in the same manner as the constituent parts of the ideal bicycle represented in figure 35. Each monocycle is constructed as follows: A steel tube B, in which a cylindrical shaft A is inserted, fits into a heavy steel ring C, and the latter is clamped to the shaft by two thumbscrews D, which pass through suitable slits in the tube. A hollow ring E is fitted on to the ring C and then encased by a second tube F. The annexed figure shows a cross-section of this mechanism. The hollow ring carries two small pegs u, which pass through suitable slits in the outer tube F. These pegs are connected in such a manner with the shaft A that by raising or lowering the latter we raise or lower respectively the ring E; we shall therefore refer to these pegs as the movable pegs. On the other hand, this raising and lowering of the shaft A shall have no effect what-

ever on the rotation of the ring E or on that of the tube F, which rotates with it, and the rotation of the ring E no effect on the other ring C.

In addition to the two movable pegs there are two fixed pegs v carried by the outer tube F. A parallelogram, similar to the centrifugal regulator of a steam engine, is constructed on each fixed and each movable peg in the manner indicated in the annexed figure.

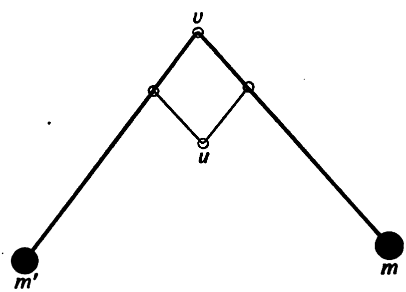

FIG. 40.

The planes of these two parallelograms are parallel to the axis of the tubes and to each other and are separated from each other only by the thickness of the tubes. Their projecting arms carry masses m and m' respectively.

The manner in which the three given monocycles are coupled together is shown in figures 41 and 42 (see next page). The steel tube B passes through the whole apparatus and is supported above and below; it is divided into three equal parts by two narrow bands G clamped to it. The tubes F rotate upon these respective parts. The frictions between the upper and lower tubes and the shaft represent the retarding forces W_1 and W_2 respectively; W_1 and W_2 would not, however, necessarily obey the laws of friction. To diminish the frictions between the middle tube and the two narrow bands

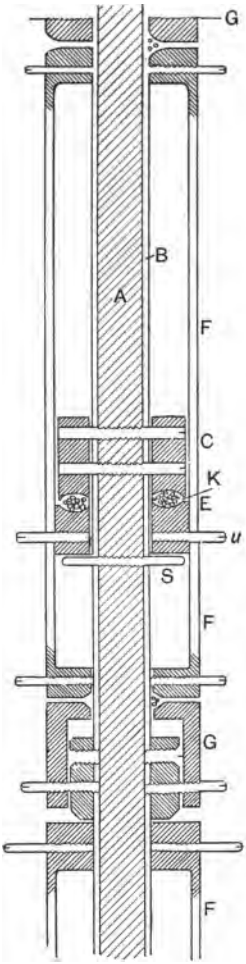

FIG. 41.

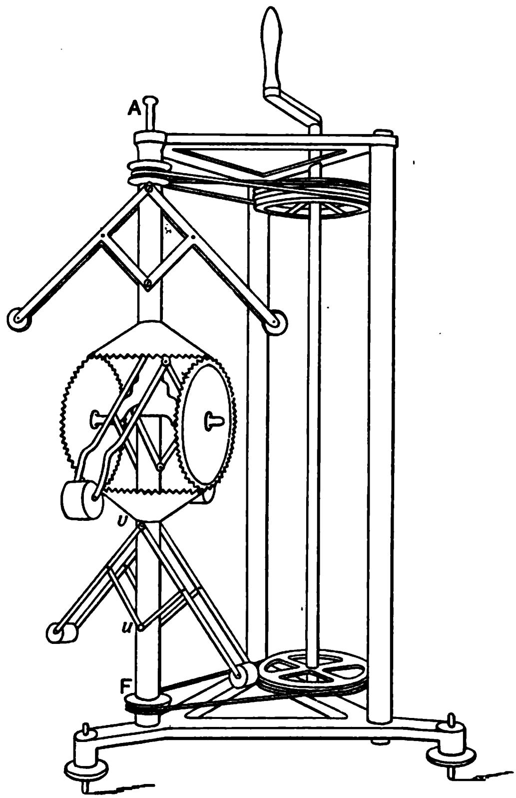

FIG. 42.

G, and also those between the fixed and rotating rings C and E respectively, small hard steel balls or shot K have been introduced into the respective sockets. The shaft A is rigidly connected with the three rings C (only the middle one is shown in figure 41), which are fitted round the steel tube B. These rings can be raised and lowered, but not turned on their mutual axis. The rotating rings E are inserted between the rings C and the pins or pegs S carried by the shaft A; these pins pass through suitable slits in the tube B. Each ring E carries the peg u of a centrifugal regulator. The upper and lower rings E and the pegs v inserted in the upper and lower tubes F carry centrifugal regulators similar to those just described, whereas the centrifugal regulators of the middle ring E and tube F are somewhat differently constructed, namely, as follows: The arms carrying the masses m and m' are secured at right angles to the two sides of a parallelogram, as indicated in the annexed figure; the similarly dotted lines represent here the

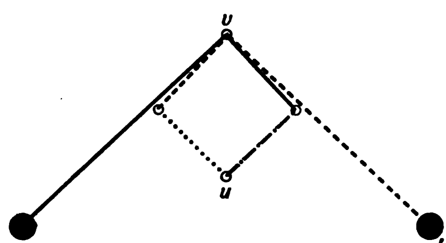

FIG. 43.

rigid parts of the system. It is evident that the masses m and m' will be drawn in towards the axis, as the pegs u and v approach each other, and that they will recede from it as the latter separate.

Since the masses carried by the arms of the middle centrifugal regulators have been chosen four times as large as

those of the upper and lower regulators, the mechanics of the given bicycle will evidently be such that

$$r_1^2 + r_3^2 = r_2^2 + r_3^2 = \text{constant},$$

r_3 decreasing as r_1 and r_2 increase and conversely (cf. p. 301). By lowering the shaft A the distance r_3 will therefore increase, and hence by formula (13) the coefficient C also, while the coefficients A and B will remain unchanged. This corresponds to an increment in the coefficient of mutual induction of two electric circuits, that is, to their approximation. The introduction of two driving points, by means of which the coefficients A and B could also be changed, has not, however, been attempted.

Two horizontal bevel-wheels, whose teeth play into those of two vertical ones, are attached to the upper and lower tubes F, see figure 42. The vertical wheels are supported by two horizontal rods P soldered to the middle tube F, and the latter pass through the centres of the vertical wheels and thus serve as axes of rotation. They correspond to the rotating discs of the ideal mechanism represented in figure 35. The lower tube F carries a horizontal wheel, and the upper tube two such; these are driven by three wheels, which can be screwed to a second vertical shaft, and the latter is turned by a handle. Lastly, the two upper pairs of wheels are connected by driving-belts in such a manner that the one pair drives the upper tube in the one direction, and the other pair this same tube in the opposite direction.

The following experiments can be performed with this apparatus: (1) The lower driving-wheel is set rotating; (a) as long as its velocity is kept constant, the upper tube F will remain at rest; as its velocity increases, the upper tube will be turned in the opposite direction, and as it decreases in the same direction; moreover, the further down the shaft A is pushed into the tube B, the more pronounced this effect or reaction;

(*b*) the shaft A is next thrust down into the tube B, and the coefficient C is thus increased; the upper tube will then be turned in the opposite direction, whereas if the shaft is suddenly raised it will receive an impulse in the same direction. (2) The upper and lower tubes F are turned in the same direction; hereby the shaft A will be pulled downwards and the coefficient C thus increased; this corresponds to an increment in the coefficient of mutual induction of two electric circuits, that is, to their approximation. (3) The upper and lower tubes are turned in opposite directions, whereby the shaft A will be thrust upwards and the coefficient C increased, etc.

The forces acting on the handles of our apparatus represent the electromotive forces that drive the primary and secondary circuits. Take the simplest case, where namely the two circuits are parallel to each other and of equal current-strengths but of opposite directions of flow, the tube carrying the mass m_3 will then receive no impulse whatever. If the coefficients of self-induction of the two circuits remain unchanged and we now increase the distances r_1 and r_2, we must then decrease the distance r_3 (cf. formula (13)) and hence the coefficient of mutual-induction C; this corresponds to a gradual separation of the given circuits and the extra energy or work thus expended to their apparent repulsion. Hereby the strings leading from the masses m_1 and m_2 and manipulated by weights attached to their free ends will be drawn up the central tube, that is, the weights (ponderable masses) will themselves be raised and hence their kinetic energy increased.

To appreciate the full value of our mechanical explanation of the given electromagnetic phenomena, let us examine somewhat more thoroughly the simple case where the angular velocities representing the current-strengths are maintained constant. The work done by the forces L_1 and L_2 brought to act on the handles of our apparatus is then

$$l_1'L_1 + l_2'L_2$$

or, by formulae (14), after putting

$$m_1 = m_2 = \frac{m_3}{4} = m,$$

$$l_1'L_1 + l_2'L_2 = 2\,m\left\{l_1'\left(l_1'r_1\frac{dr_1}{dt} + l_1'r_3\frac{dr_3}{dt} + l_2'r_3\frac{dr_3}{dt}\right)\right.$$

$$\left. + l_2'\left(l_2'r_2\frac{dr_2}{dt} + l_2'r_3\frac{dr_3}{dt} + l_1'r_3\frac{dr_3}{dt}\right)\right\}dt$$

$$= 2\,m\left\{l_1'^2r_1\frac{dr_1}{dt} + l_2'^2r_2\frac{dr_2}{dt} + (l_1' + l_2')^2r_3\frac{dr_3}{dt}\right\}dt.$$

The total work done in raising the three weights attached to the free ends of the strings leading from the masses m_1, m_2 and m_3 is evidently

$$-(R_1dr_1 + R_2dr_2 + R_3dr_3)$$

or, by formulae (14),

$$-(R_1dr_1 + R_2dr_2 + R_3dr_3)$$
$$= m\left\{l_1'^2r_1\frac{dr_1}{dt} + l_2'^2r_2\frac{dr_2}{dt} + (l_1 + l_2)^2r_3\frac{dr_3}{dt}\right\}dt,$$

where as above we have put $m_1 = m_2 = \frac{m_3}{4} = m$.

It follows therefore that the forces that drive the handles of our apparatus do here twice as much work as is necessary to raise the given weights; the other half is expended in increasing its kinetic energy T, that is, that of the masses m_1, m_2 and m_3, for write

$$m_1 = m_2 = \frac{m_3}{4} = m$$

in the given expression (14) for T, and we get

$$T = \frac{m}{2}\{l_1'^2r_1^2 + l_2'^2r_2^2 + (l_1' + l_2')^2r_3^2\},$$

hence $$dT = m\left\{l_1'^2r_1\frac{dr_1}{dt} + l_2'^2r_2\frac{dr_2}{dt} + (l_1' + l_2')^2r_3\frac{dr_3}{dt}\right\}dt;$$

this energy corresponds to the invisible (kinetic) ether-energy.

We have now designated the following quantity in the first chapter as a potential energy (cf. formula (7, II.)):

$$V=\frac{D}{8\pi}(P^2+Q^2+R^2);$$

if, however, stationary electric currents can be regarded as cyclic motions defined by cyclic coordinates $l_1, l_2, \ldots$, whose rates of change $l_1', l_2', \ldots$ represent their current-strengths, we must then also be able to write V in the following form—we shall examine here only two electric circuits:

$$V=\frac{A}{2}l_1'^2+\frac{B}{2}l_2'^2+Cl_1'l_2', \qquad (15)$$

where A, B, and C are functions of the slowly changing parameters. The above principle must then also hold here, namely, that the work done during any displacement of an electric circuit of constant current-strength by the forces that act apparently at a distance between it and any other circuit is equal to the increment of the medium-energy V_{12} arising from that displacement. We have now already found expressions for this medium-energy (cf. formulae (26, XII.) and (10, XIII.)); their validity is of course entirely independent of any assumption concerning either the existence or non-existence of real magnetism; as, however, both the fundamental principles and the mechanical analogies of Chapter I. exclude the very conception of real magnetism, we shall accept the latter assumption henceforth.

The increment of the medium-energy (cf. formula (10, XIII.)) is usually known as the work done by the forces that act apparently at a distance; the laws of induced currents follow directly from it.

If one of the above circuits is replaced by a solenoid with its negative end removed to infinite distance, we find by formulae (7, 8 and 12, XII.)—their validity is

also independent of any assumption concerning the existence of real magnetism—the following values for α, β, γ in any point at the finite distance ρ from its positive end:

$$\alpha = -\frac{i_1 fN}{\mathfrak{B}}\frac{d\left(\frac{1}{\rho}\right)}{dx}, \quad \beta = -\frac{i_1 fN}{\mathfrak{B}}\frac{d\left(\frac{1}{\rho}\right)}{dy}, \quad \gamma = -\frac{i_1 fN}{\mathfrak{B}}\frac{d\left(\frac{1}{\rho}\right)}{dz}.$$

If we take $\zeta = \frac{4\pi i_1 fMN}{\mathfrak{B}}$ as the number of lines of induction that radiate from the positive end of the given solenoid, the number of lines that pass through unit-surface at right angles to their direction of flow at the distance ρ from the given end will evidently be given by the value of the vector

$$\frac{i_1 fMN}{\mathfrak{B}\rho^2} = M\sqrt{\alpha^2+\beta^2+\gamma^2}.$$

The total energy V_{12} arising from the combined presence of any solenoid and an electric circuit of current-strength i is now given by formula (26, XII.), namely,

$$V_{12} = -\frac{i}{\mathfrak{B}}\int M do[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]$$
$$= -\frac{i}{\mathfrak{B}}\Sigma Z,$$

where ΣZ denotes the number of lines of induction sent through the circuit by the positive end of the solenoid. If Ω denotes the solid angle subtended by the circuit at the given end, then

$$\Sigma Z = \zeta\frac{\Omega}{4\pi} = \frac{i_1 fMN}{\mathfrak{B}}\Omega,$$

hence $$V_{12} = -\frac{ii_1 fMN}{\mathfrak{B}}\Omega = -i_m i'_m fMN\Omega.$$

As this is the total energy arising from the combined presence of our circuit and solenoid, it must correspond to the quantity $Cl_1'l_2'$ of the above expression (15) for V.

Since the negative differential of this quantity with regard to any coordinate (x, y, z) always gives the force that must be brought to act on that coordinate to keep it constant, it follows that its positive derivative is the force arising from the combined action of both circuit and solenoid tending to increase that coordinate. These derivatives are now the very expressions that, under the assumption of real magnetism, represented the forces, which an electric circuit of current-strength i and a magnetic pole of intensity $m_r = i_{1m}fMN$ exercised on each other; this follows directly from formulae (13 and 15, XII.), which give

$$X = m_r\frac{d\psi}{dx} = -m_r\frac{d}{dx}\left(\frac{i\Omega}{\mathfrak{B}}\right) = -\frac{ii_1fMN}{\mathfrak{B}^2}\frac{d\Omega}{dx},$$

$$Y = -\frac{ii_1fMN}{\mathfrak{B}^2}\frac{d\Omega}{dy} \text{ and } Z = -\frac{ii_1fMN}{\mathfrak{B}^2}\frac{d\Omega}{dz}.$$

Similarly, we could show that the ends of two solenoids of current-strengths i and i_1 act on each other like magnets of intensities i_mfMN and $i_{1m}fMN_1$ respectively.

We see therefore that we can explain all magnetic phenomena, even if we deny the existence of real magnetism, provided we only replace permanent magnets by solenoids, and that we can thus retain our mechanical view or fundament presented in Chapter I. Two apparent differences should, however, be noted. In the first place, we have already observed, when electric circuits or solenoids are displaced, that the electromotive forces do twice as much work as is necessary to bring about the given displacement, but that this is not true of the action between electric circuits and magnets or magnets alone. This apparent disparity can be explained in the following manner: It is, namely, natural to suppose that the electromotive forces that maintain the molecular currents of a magnet are included among the unknown molecular forces themselves. The energy derived from them cannot thus arise from visible

heads of energy, but it must be regarded as belonging to the (molecular) medium-energy. Hence, if an electric circuit and magnet approach each other and the visible work Q_1 is done, the electromotive forces will do not only the work $Q_2 = Q_1$, but also an equal amount of work $Q_3 = Q_1$, provided the current-strength of the circuit and the intensity of the magnet remain unchanged. This corresponds to an amount of work Q_1 done and to the creation of an equal amount Q_4 of medium-energy. Since the work done by the electromotive forces that drive the molecular currents is now supposed to be derived directly from the medium-energy and the latter is in turn increased by the same amount Q_4 of medium-energy, it is most natural to assume that only the electromotive forces of the circuit do the visible work Q_1, whereas the medium-energy thereby remains unaltered. It thus follows that, when two magnets approach each other, the work done must be derived from the medium-energy, since no visible electromotive forces are present. Secondly, we have seen on p. 233 that the action between magnets immersed in a fluid is inversely proportional to its magnetic conductivity M and on p. 275 that that between solenoids is directly proportional to this constant, but, in the latter case, with the restriction that M is constant throughout the given region; such a condition could of course only be satisfied when the entire region between and within the molecular currents of the solenoid were filled with the same fluid as that in which the solenoid itself is immersed, a condition which is of course never realized in a steel magnet.

CHAPTER XV.

SECTION XXXIV. INDUCTION; ANOTHER FORM OF OUR FUNDAMENTAL EQUATIONS. THE INDUCTION IN A CLOSED CIRCUIT; NEUMANN'S LAWS AND THOMSON'S EQUATIONS.

THE above investigations, from Chapter V. on, pertain chiefly to the stationary motions of the ether, that is, to first approximations or integrals of first approximations of our general equations (9, II.) and (10, II.). Let us now examine their other integrals, second and higher approximations, those namely which do not represent stationary motions. The importance of these integrals becomes apparent, when we realize that the phenomena of induction of not only variable electric currents but electric oscillations—the former are to be regarded as only forming a transition from the stationary to the oscillatory state of the ether —must be determined from them. We shall distinguish in the following between two classes: those phenomena, to which second and higher approximations of our fundamental equations give rise, shall constitute the first class; while the electric oscillations or waves, which cannot be expressed by such series, shall be included in the second class. To illustrate this distinction let us briefly recall the investigations of § 10 on the vibrations of the elastic band. We have seen there that, when the free end of the elastic band was set in slow transverse vibrations, the first approximation for the vertical

displacement y of any particle of it at the distance x from its fixed end was

$$y = \frac{x}{l} f(t)$$

(cf. formula (6, V.)). As this equation represents a straight line, the band is here uniformly stretched; it corresponds to the stationary state of the ether. A second approximation for y was

$$y = \frac{x}{l} f(t) + \frac{x}{6a^2}\left(\frac{x^2}{l} - l^2\right) f''(t)$$

(cf. formula (7, V.)). The second term of this expression is the correction that must be applied to the first approximation for y. The corresponding term or correction in electricity gives rise to the phenomena of induction. Similarly corrections of higher orders give inductions of higher orders.

To obtain second and higher approximations we write our fundamental equations as follows:

$$\nabla^2 P = \frac{1}{\mathfrak{V}} \frac{d}{dt}\left[\frac{d(M\beta)}{dz} - \frac{d(M\gamma)}{dy}\right] + \frac{d}{dx}\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}\right), \quad \ldots .(1)$$

with similar expressions in Q and R.

These equations evidently give

$$P = P_1 + P_2, \quad Q = Q_1 + Q_2, \quad R = R_1 + R_2, \quad \ldots\ldots .(2)$$

$$\left.\begin{aligned} \text{where} \quad P_1 &= -\frac{1}{4\pi\mathfrak{V}} \int \frac{d}{dt}\left[\frac{d(M\beta)}{dz'} - \frac{d(M\gamma)}{dy'}\right]\frac{d\tau'}{\rho} \\ \text{and} \quad P_2 &= -\frac{1}{4\pi} \int \frac{d}{dx'}\left(\frac{dP}{dx'} + \frac{dQ}{dy'} + \frac{dR}{dz'}\right)\frac{d\tau'}{\rho} \end{aligned}\right\}, \quad \ldots\ldots(3)$$

with similar expressions for Q_1, Q_2 and R_1, R_2. Here x, y, z denote the point at which the values of P, Q, R are sought, x', y', z' any point of space and ρ the distance between these points.

The integral

$$\int \frac{d}{dt}\left[\frac{d(M\beta)}{dz'}\right]\frac{d\tau'}{\rho} = \int d\left[\frac{d(M\beta)}{dt}\right]\frac{dx'dy'}{\rho},$$

integrated by parts, gives

$$\iint dx'dy'\left|\frac{\frac{d}{dt}(M\beta)}{\rho}\right|_{-\infty}^{+\infty}-\int\frac{d}{dt}(M\beta)dx'dy'd\left(\frac{1}{\rho}\right).$$

The first of these integrals vanishes in conformity to the principle of the continuity of transitions, whereas the second can be written in the following form by the relation $\dfrac{d\left(\frac{1}{\rho}\right)}{dz'}=-\dfrac{d\left(\frac{1}{\rho}\right)}{dz}$:

$$\int\frac{d}{dt}(M\beta)\,d\tau'\frac{d\left(\frac{1}{\rho}\right)}{dz}=\frac{d^2}{dtdz}\int M\beta\frac{d\tau'}{\rho}.$$

The other integrals of formulae (3) can be similarly treated.

If, as in § 26, we put

$$\int M\alpha\frac{d\tau'}{\rho}=\overline{M\alpha},\text{ etc. and}$$

$$\frac{1}{4\pi}\int\left(\frac{dP}{dx'}+\frac{dQ}{dy'}+\frac{dR}{dz'}\right)\frac{d\tau'}{\rho}=\frac{1}{4\pi}\overline{\left(\frac{dP}{dx'}+\frac{dQ}{dy'}+\frac{dR}{dz'}\right)}=\phi,\ldots(4)$$

we can thus write formulae (3) as follows:

$$P_1=-\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{d(\overline{M\beta})}{dz}-\frac{d(\overline{M\gamma})}{dy}\right]\text{ and }P_2=-\frac{d\phi}{dx},\ldots(5)$$

with similar expressions for Q_1, Q_2 and R_1, R_2.

We have therefore, by formulae (2), the following values for P, Q, R:

$$\left.\begin{aligned}P&=-\frac{d\phi}{dx}+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{d(\overline{M\gamma})}{dy}-\frac{d(\overline{M\beta})}{dz}\right]\\Q&=-\frac{d\phi}{dy}+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{d(\overline{M\alpha})}{dz}-\frac{d(\overline{M\gamma})}{dx}\right]\\R&=-\frac{d\phi}{dz}+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{d(\overline{M\beta})}{dx}-\frac{d(\overline{M\alpha})}{dy}\right]\end{aligned}\right\}\ldots\ldots\ldots(6)$$

The exact expressions for α, β, γ can be found in a similar manner. Instead of equations (1) we have, namely, the analogous ones

$$\nabla^2\alpha=\frac{4\pi}{\mathfrak{B}}\left\{\frac{d}{dy}\left[L(R+Z)+\frac{D}{4\pi}\frac{dR}{dt}\right]-\frac{d}{dz}\left[L(Q+Y)+\frac{D}{4\pi}\frac{dQ}{dt}\right]\right\}$$
$$+\frac{d}{dx}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right),$$

with similar equations in β and γ; and these give

$$\left.\begin{aligned}\alpha&=-\frac{d\psi}{dx}+\frac{1}{\mathfrak{B}}\left\{\frac{d}{dz}\left[\overline{L(Q+Y)+\frac{D}{4\pi}\frac{dQ}{dt}}\right]-\frac{d}{dy}\left[\overline{L(R+Z)+\frac{D}{4\pi}\frac{dR}{dt}}\right]\right\}\\ \beta&=-\frac{d\psi}{dy}+\frac{1}{\mathfrak{B}}\left\{\frac{d}{dx}\left[\overline{L(R+Z)+\frac{D}{4\pi}\frac{dR}{dt}}\right]-\frac{d}{dz}\left[\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}\right]\right\}\\ \gamma&=-\frac{d\psi}{dz}+\frac{1}{\mathfrak{B}}\left\{\frac{d}{dy}\left[\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}\right]-\frac{d}{dx}\left[\overline{L(Q+Y)+\frac{D}{4\pi}\frac{dQ}{dt}}\right]\right\}\end{aligned}\right\},\tag{7}$$

where

$$\left.\begin{aligned}&\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}=\int\frac{d\tau'}{\rho}\left[L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}\right],\text{ etc.}\\ &\text{and }\psi=\frac{1}{4\pi}\left(\overline{\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}}\right)=\frac{1}{4\pi}\int\frac{d\tau'}{\rho}\left(\frac{d\alpha}{dx'}+\frac{d\beta}{dy'}+\frac{d\gamma}{dz'}\right)\end{aligned}\right\}.\tag{8}$$

ψ is here the same quantity as ψ, the magnetic potential, of § 26.

Formulae (6) show that, when the magnetic polarization of the field changes sufficiently rapidly, P, Q, R can no longer be regarded as the partial derivatives of one and the same quantity ϕ with regard to the coordinates; it follows, moreover, from formulae (20 and 25, XI.) that the corrections to be applied to the electric forces P, Q, R arising from such variations are analogous to those already found for the magnetic forces α, β, γ due to the presence of stationary electric currents. We can thus introduce here a new feature into our

concrete representation and assume that the forces exerted by variable magnets or, in fact, those arising from any variations in the magnetic polarization of the field, on quantities of electricity and also on one another are analogous to the forces, which stationary electric currents exercise on magnetic poles or on one another.

Equations (6) and (7) contain two new unknown quantities ϕ and ψ—formulae (4) and (8) are merely notations. To obtain a complete system of independent equations for the determination of the eight quantities P, Q, R, α, β, γ, ϕ, ψ we make use of the two following relations in addition to the six given equations:

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]$$
$$+\frac{dL(P+X)}{dx}+\frac{dL(Q+Y)}{dy}+\frac{dL(R+Z)}{dz}=0 \quad \ldots.(9)$$

(cf. equation (2, III.)) and

$$\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}=0\,; \quad \ldots\ldots..(10)$$

in assuming the latter of which we are excluding all real magnetism from the field. These eight equations together with the condition that ϕ and ψ vanish at infinite distance evidently determine the eight given quantities uniquely, and they can thus be regarded as our fundamental equations in place of our former ones (9, II.) and (10, II.). For surfaces of discontinuity the above equations and integrals must of course be replaced by those derived from the given expressions by the application of the principle of the continuity of transitions.

Before examining the new fundamental equations (6), (7), (9) and (10) in their general form (cf. next article), let us consider their application to the simple

but important problem of determining the induction in any given closed circuit s placed in a field produced by electric currents. It is immaterial whether a current is flowing through the circuit s or not. If P, Q, R denote the components of the electromotive force at any point (x, y, z) of the given circuit and ds any linear element of it, the expression

$$[P\cos(ds, x) + Q\cos(ds, y) + R\cos(ds, z)]ds$$
$$= Pdx + Qdy + Rdz$$

will evidently represent the electromotive force acting in that element. The total electromotive force acting in the given circuit s will thus be

$$\int(Pdx + Qdy + Rdz), \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where the integration is extended round the circuit s.

The current induced in any circuit is now given by the last terms of formulae (6). To eliminate the function ϕ from these equations we form the expression (11); replacing there P, Q, R by their values (6) we have

$$\int(Pdx + Qdy + Rdz) = \frac{1}{4\pi\vartheta}\frac{d}{dt}\int\left[\left(\frac{d\overline{M\gamma}}{dy} - \frac{d\overline{M\beta}}{dz}\right)dx\right.$$
$$\left. + \left(\frac{d\overline{M\alpha}}{dz} - \frac{d\overline{M\gamma}}{dx}\right)dy + \left(\frac{d\overline{M\beta}}{dx} - \frac{d\overline{M\alpha}}{dy}\right)dz\right],$$

in forming which we observe that the following integral has vanished:

$$\int\left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz\right) = \int d\phi.$$

We can write the second integral of this equation by Stokes' theorem (1, XIII.) as follows:

$$\int(Pdx+Qdy+Rdz)$$

$$=\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\int\left\{\left[\frac{d}{dy}\left(\frac{d\overline{M\beta}}{dx}-\frac{d\overline{M\alpha}}{dy}\right)-\frac{d}{dz}\left(\frac{d\overline{M\alpha}}{dz}-\frac{d\overline{M\gamma}}{dx}\right)\right]\cos(n,x)\right.$$

$$+\left[\frac{d}{dz}\left(\frac{d\overline{M\gamma}}{dy}-\frac{d\overline{M\beta}}{dz}\right)-\frac{d}{dx}\left(\frac{d\overline{M\beta}}{dx}-\frac{d\overline{M\alpha}}{dy}\right)\right]\cos(n,y)$$

$$\left.+\left[\frac{d}{dx}\left(\frac{d\overline{M\alpha}}{dz}-\frac{d\overline{M\gamma}}{dx}\right)-\frac{d}{dy}\left(\frac{d\overline{M\gamma}}{dy}-\frac{d\overline{M\beta}}{dz}\right)\right]\cos(n,z)\right\}do$$

$$=-\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\int\left\{\left[\nabla^2\overline{M\alpha}-\frac{d}{dx}\left(\frac{d\overline{M\alpha}}{dx}+\frac{d\overline{M\beta}}{dy}+\frac{d\overline{M\gamma}}{dz}\right)\right]\cos(n,x)\right.$$

$$+\left[\nabla^2\overline{M\beta}-\frac{d}{dy}\left(\frac{d\overline{M\alpha}}{dx}+\frac{d\overline{M\beta}}{dy}+\frac{d\overline{M\gamma}}{dz}\right)\right]\cos(n,y)$$

$$\left.+\left[\nabla^2\overline{M\gamma}-\frac{d}{dz}\left(\frac{d\overline{M\alpha}}{dx}+\frac{d\overline{M\beta}}{dy}+\frac{d\overline{M\gamma}}{dz}\right)\right]\cos(n,z)\right\}do$$

or, by formula (10), which gives

$$\frac{d\overline{M\alpha}}{dx}+\frac{d\overline{M\beta}}{dy}+\frac{d\overline{M\gamma}}{dz}=0,$$

and by the relations

$$\nabla^2\overline{M\alpha}=-4\pi M\alpha,\ \nabla^2\overline{M\beta}=-4\pi M\beta \text{ and } \nabla^2\overline{M\gamma}=-4\pi M\gamma,$$

as follows:

$$\int(Pdx+Qdy+Rdz)$$

$$=\frac{1}{\mathfrak{V}}\frac{d}{dt}\int M[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]do. \quad (12)$$

Hertz obtains this equation directly by applying Stokes' theorem to the initial expression (11) for the

total electromotive force, namely,

$$\int(P\,dx+Q\,dy+R\,dz)=\int\Big[\Big(\frac{dR}{dy}-\frac{dQ}{dz}\Big)\cos(n,x)$$

$$+\Big(\frac{dP}{dz}-\frac{dR}{dx}\Big)\cos(n,y)+\Big(\frac{dQ}{dx}-\frac{dP}{dy}\Big)\cos(n,z)\Big\}do,$$

and by replacing the given curls in P, Q, R by their respective values from equations (10, II.). We have chosen the above somewhat more complicated derivation for the purpose of throwing as much light as possible on the nature of the quantities

$$\frac{1}{4\pi\mathfrak{B}}\,\frac{d}{dt}\Big(\frac{d\overline{M\gamma}}{dy}-\frac{d\overline{M\beta}}{dz}\Big),\quad \frac{1}{4\pi\mathfrak{B}}\,\frac{d}{dt}\Big(\frac{d\overline{M\alpha}}{dz}-\frac{d\overline{M\gamma}}{dx}\Big),$$

$$\frac{1}{4\pi\mathfrak{B}}\,\frac{d}{dt}\Big(\frac{d\overline{M\beta}}{dx}-\frac{d\overline{M\alpha}}{dy}\Big),$$

the corrections to be applied to the quantities P, Q, R of stationary flow; we observe, namely, that only the latter quantities vanish when integrated round the given circuit.

If σ denotes any cross-section of the circuit s and ω its current-density, $\omega\sigma$ will be the quantity of electricity that flows through that cross-section during unit-time, that is, its current-strength; this quantity is now given by formula (6, III.), namely,

$$I=\sigma LN=\sigma L[P\cos(n,x)+Q\cos(n,y)+R\cos(n,z)].$$

This formula gives

$$P\,dx+Q\,dy+R\,dz=\frac{I\,ds}{\sigma L}\quad\ldots\ldots\ldots\ldots\ldots(13)$$

or, integrated round the given circuit,

$$\int(P\,dx+Q\,dy+R\,dz)=\int\frac{I\,ds}{\sigma L};$$

it can be written by formula (12) as follows:

$$\int\frac{I\,ds}{\sigma L}=\int(P\,dx+Q\,dy+R\,dz)$$
$$=\frac{1}{\mathfrak{V}}\frac{d}{dt}\int M[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]do. \quad (14)$$

The last integral of this formula will evidently be an approximately linear function of the time for all disturbances of the ether except its rapid oscillations; the induced current will thus be approximately stationary and hence its current-strength I approximately the same throughout the whole circuit. We can therefore write

$$I\int\frac{ds}{\sigma L}=\int(P\,dx+Q\,dy+R\,dz)$$
$$=\frac{1}{\mathfrak{V}}\frac{d}{dt}\int M[\alpha\cos(n,x)+\beta\cos(n,y)+\gamma\cos(n,z)]do \quad (15)$$

or by formula (26, XII.), which holds for any field produced by electric currents only,

$$I\int\frac{ds}{\sigma L}=-\frac{d}{dt}\left(\frac{V_{12}}{I}\right), \quad\text{..................(16)}$$

where V_{12} denotes the potential between the given circuit and the currents of the field. Neumann's well-known laws for galvanic induction follow directly from this potential (V_{12}); if, namely, the given field is produced by only one circuit s' of current-strength i', V_{12} will be given by formula (10, XIII.), and the above expression (16) will then assume the special form

$$I\int\frac{ds}{\sigma L}=-\frac{d}{dt}\left\{\frac{i'M}{\mathfrak{V}^2}\iint\frac{\cos(ds,ds')\,ds\,ds'}{\rho}\right\} \quad\text{.......(17)}$$

or, if we exclude all displacements and deformations of the circuit s', the following:

$$I\int\frac{ds}{\sigma L}=-B\frac{di'}{dt}, \quad\text{....................(18)}$$

where
$$B=\frac{M}{\mathfrak{V}^2}\iint\frac{\cos(ds,ds')\,ds\,ds'}{\rho}. \quad\text{.............(19)}$$

In our above assumption, that the last integral of formula (14) is an approximately linear function of the time and hence the induced current approximately stationary, we are also neglecting the self-induction of the induced current arising from any variation in its current-strength I due to variations in the rate of change in the intensity of the field. In taking this correction into consideration we should have to conceive the given circuit s as composed of an infinite number of current-threads and then determine their mutual induction; this would evidently be given by a similar expression to that above (16), provided all displacements of the given circuit (s) were excluded, and we should thus have

$$I\int\frac{ds}{\sigma L}=-\frac{d}{dt}\left(\frac{V_{12}}{I}\right)-A\frac{dI}{dt}.$$

If the field were produced by a second circuit s' of current-strength i', we could write this expression by formulae (18) and (19) as follows:

$$I\int\frac{ds}{\sigma L}+A\frac{dI}{dt}+B\frac{di'}{dt}=0;$$

a similar equation would of course hold for the second circuit s' taken as secondary circuit, namely,

$$i'\int\frac{ds}{\sigma' L'}+A\frac{di'}{dt}+B\frac{dI}{dt}=0.$$

If external electromotive forces, as hydro- or thermo-electromotive forces, reside in the given circuits, we should evidently have the two following equations instead of the above pair:

$$\left.\begin{aligned} I\int\frac{ds}{\sigma L}+A\frac{dI}{dt}+B\frac{di'}{dt}=\epsilon \\ i'\int\frac{ds}{\sigma' L'}+A\frac{di'}{dt}+B\frac{dI}{dt}=\epsilon' \end{aligned}\right\}, \quad\text{..............(20)}$$

where ϵ and ϵ' denote the external electromotive forces that reside in the circuits s and s' respectively. These correction-formulae for galvanic induction were first established by Sir William Thomson (Lord Kelvin). The actual determination of the integrals A and B is, of course, often difficult and laborious.

SECTION XXXV. MAXWELL'S EQUATIONS OF ACTION AT A DISTANCE. EXTENSION OF OUR CONCRETE REPRESENTATION; A SYSTEM IN WHICH $L=X=Y=Z=0$. CHARACTERISTIC FORM OF MAXWELL'S EQUATIONS FOR A RING.

We have seen in the preceding article that the six fundamental equations (9, II.) and (10, II.) can be replaced by the eight equations (6), (7), (9) and (10) and that the latter may thus be regarded as fundamental equations instead of the former. In these new equations the quantities P, Q, R and α, β, γ are now expressed as functions, not only of the state in the respective neighbouring volume-elements but of integrals extended to all volume-elements of space—we could therefore distinguish these fundamental equations from the original ones by designating them as Maxwell's equations of action at a distance.* For this reason they are not so logical as our original fundamental equations; they are also less applicable to the determination of rapid electric oscillations, since the expressions for α, β, γ require a knowledge of the values of P, Q, R and conversely those for P, Q, R the values of α, β, γ at *every* point of space. On the other hand, they are exactly suited for the determination of approximately stationary or aphotic disturbances. Take for example a finite number of linear electric circuits; we can characterize the total electromotive force acting in any circuit approximately

* Cf. footnote p. 388.

by a single variable I, its current-strength; and first determine P, Q, R under the supposition that the given current is stationary; we have then by formulae (13):

$$P=\frac{I\cos(n,x)}{\sigma L},\quad Q=\frac{I\cos(n,y)}{\sigma L},\quad R=\frac{I\cos(n,z)}{\sigma L}.\ \ldots(21)$$

We next substitute these values for P, Q, R in equations (7), and we determine α, β, γ as functions of I and ψ as follows:

$$\alpha(x_1, y_1, z_1)=-\frac{d\psi}{dx_1}$$
$$+\frac{1}{\mathfrak{B}}\left\{\frac{d}{dz_1}\int\frac{d\tau_1'}{\rho}\left[\frac{I\cos(n,y)}{\sigma}+LY+\frac{D}{4\pi}\frac{d}{dt}\left(\frac{I\cos(n,y)}{\sigma L}\right)\right]\right.$$
$$\left.-\frac{d}{dy_1}\int\frac{d\tau_1'}{\rho}\left[\frac{I\cos(n,z)}{\sigma}+LZ+\frac{D}{4\pi}\frac{d}{dt}\left(\frac{I\cos(n,z)}{\sigma L}\right)\right]\right\},$$

where $\quad \rho^2=(x_1'-x_1)^2+(y_1'-y_1)^2+(z_1'-z_1)^2.$

If we assume that the relative position of both the circuits and their elements remains unchanged, we can evidently write this expression for α as follows:

$$\alpha(x_1, y_1, z_1)$$
$$=-\frac{d\psi}{dx_1}+\frac{1}{\mathfrak{B}}\left\{\sum_{i=1}^{i=i}I_i\int_i\frac{d\tau_1'}{\sigma}\left[\cos(n,y)\frac{d}{dz_1}\left(\frac{1}{\rho}\right)-\cos(n,z)\frac{d}{dy_1}\left(\frac{1}{\rho}\right)\right]\right.$$
$$+\frac{1}{4\pi}\sum_{i=1}^{i=i}\frac{dI_i}{dt}\int_i\frac{Dd\tau_1'}{\sigma L}\left[\cos(n,y)\frac{d}{dz_1}\left(\frac{1}{\rho}\right)-\cos(n,z)\frac{d}{dy_1}\left(\frac{1}{\rho}\right)\right]$$
$$\left.+\int Ld\tau_1'\left[Y\frac{d}{dz_1}\left(\frac{1}{\rho}\right)-Z\frac{d}{dy_1}\left(\frac{1}{\rho}\right)\right]\right\}, \ldots\ldots\ldots\ldots(22)$$

where the index i suffixed to the integration-signs denotes that the given integrations are to be extended only to the volume-elements $d\tau_1'$ of the given circuit; similar expressions hold for Q and R.

Lastly, we replace α, β, γ in formulae (6) by these

values (22), and we find the following expressions for P, Q, R at any point (x, y, z) of any circuit k:

$$
\begin{aligned}
P_k(x, y, z) = &-\frac{d\phi}{dx} - \frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\iiint M\frac{d}{dy}\left(\frac{1}{\rho}\right)dx'dy'd\psi \\
&+\frac{1}{4\pi\mathfrak{V}^2}\sum_{i=1}^{i=i}\frac{dI_i}{dt}\int M\frac{d}{dy}\left(\frac{1}{\rho}\right) \\
&\quad\times\left\{\int_i\frac{d\tau''}{\sigma}\left[\cos(n, x)\frac{d}{dy'}\left(\frac{1}{\rho'}\right)-\cos(n, y)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau' \\
&+\frac{1}{(4\pi\mathfrak{V})^2}\sum_{i=1}^{i=i}\frac{d^2I_i}{dt^2}\int M\frac{d}{dy}\left(\frac{1}{\rho}\right) \\
&\quad\times\left\{\int_i\frac{Dd\tau''}{\sigma L}\left[\cos(n, x)\frac{d}{dy'}\left(\frac{1}{\rho'}\right)-\cos(n, y)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau \\
&+\frac{1}{4\pi\mathfrak{V}^2}\frac{d}{dt}\int M\frac{d}{dy}\left(\frac{1}{\rho}\right)\left\{\int Ld\tau''\left[X\frac{d}{dy'}\left(\frac{1}{\rho'}\right)-Y\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau' \\
&+\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\iiint M\frac{d}{dz}\left(\frac{1}{\rho}\right)dx'dz'd\psi \\
&-\frac{1}{4\pi\mathfrak{V}^2}\sum_{i=1}^{i=i}\frac{dI_i}{dt}\int M\frac{d}{dz}\left(\frac{1}{\rho}\right) \\
&\quad\times\left\{\int_i\frac{d\tau''}{\sigma}\left[\cos(n, z)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)-\cos(n, x)\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau' \\
&-\frac{1}{(4\pi\mathfrak{V})^2}\sum_{i=1}^{i=i}\frac{d^2I_i}{dt^2}\int M\frac{d}{dz}\left(\frac{1}{\rho}\right) \\
&\quad\times\left\{\int_i\frac{Dd\tau''}{\sigma L}\left[\cos(n, z)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)-\cos(n, x)\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau' \\
&-\frac{1}{4\pi\mathfrak{V}^2}\frac{d}{dt}\int M\frac{d}{dz}\left(\frac{1}{\rho}\right) \\
&\quad\times\left\{\int Ld\tau''\left[Z\frac{d}{dx'}\left(\frac{1}{\rho'}\right)-X\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\right\}d\tau' \quad\text{......(23)}
\end{aligned}
$$

and similar expressions for Q and R, where

$$\rho^2=(x'-x)^2+(y'-y)^2+(z'-z)^2$$

and $$\rho'^2=(x''-x')^2+(y''-y')^2+(z''-z')^2;$$

the arbitrary point (x_1, y_1, z_1) of formulae (22) has become here the special point (x', y', z'), whereas the former vector ρ, which radiated from the point (x_1, y_1, z_1), has been replaced by the vector ρ'—ρ' radiates from the special point—and the former volume-element $d\tau_1'$ by $d\tau''$.

By partial integration the second and sixth terms of the above expression for P can be written as follows:

$$-\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\iiint M\frac{d}{dy}\left(\frac{1}{\rho}\right)dx'dy'd\psi$$
$$=\frac{1}{4\pi\mathfrak{B}}\int\frac{d}{dz'}\left[M\frac{d}{dy}\left(\frac{1}{\rho}\right)\right]\frac{d\psi}{dt}d\tau'$$

and $$\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\iiint M\frac{d}{dz}\left(\frac{1}{\rho}\right)dx'dz'd\psi$$
$$=-\frac{1}{4\pi\mathfrak{B}}\int\frac{d}{dy'}\left[M\frac{d}{dz}\left(\frac{1}{\rho}\right)\right]\frac{d\psi}{dt}d\tau'.$$

If M is constant with regard to x, y, z, these two integrals evidently cancel each other, since

$$\frac{d}{dz'}\left[M\frac{d}{dy}\left(\frac{1}{\rho}\right)\right]=\frac{d}{dy'}\left[M\frac{d}{dz}\left(\frac{1}{\rho}\right)\right],$$

and the magnetic potential ψ thus disappears from our expression (23) for P and similarly from those for Q and R. Although these integrals do not vanish in media, for which M is a function of x, y, z, they will in general assume values that are so small in comparison to those of the other terms of the given expressions for P, Q, R, that they can be rejected; at all events, as the present development is only an approximate one, the rejection of such terms will only correspond to an approximation of a lower order.

We can thus write the above expressions (23) for P, Q, R as follows:

$$\left.\begin{aligned} P_k(x, y, z) &= -\frac{d\phi}{dx} + \sum_{i=1}^{i=i} A_{ik}\frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik}\frac{d^2I_i}{dt^2} + C_k \\ Q_k(x, y, z) &= -\frac{d\phi}{dy} + \sum_{i=1}^{i=i} A_{ik}'\frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik}'\frac{d^2I_i}{dt^2} + C_k' \\ R_k(x, y, z) &= -\frac{d\phi}{dz} + \sum_{i=1}^{i=i} A_{ik}''\frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik}''\frac{d^2I_i}{dt^2} + C_k'' \end{aligned}\right\}, \ldots (24)$$

where

$$\left.\begin{aligned} A_{ik} = \frac{1}{4\pi\mathfrak{V}^2}\int Md\tau'\Big\{&\frac{d}{dy}\left(\frac{1}{\rho_k}\right)\int_i \frac{d\tau''}{\sigma} \\ &\times\left[\cos(n, x)\frac{d}{dy'}\left(\frac{1}{\rho'}\right) - \cos(n, y)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right] \\ &-\frac{d}{dz}\left(\frac{1}{\rho_k}\right)\int_i \frac{d\tau''}{\sigma}\left[\cos(n, z)\frac{d}{dx'}\left(\frac{1}{\rho'}\right) - \cos(n, x)\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\Big\} \\ B_{ik} = \frac{1}{(4\pi\mathfrak{V})^2}\int Md\tau'\Big\{&\frac{d}{dy}\left(\frac{1}{\rho_k}\right)\int_i \frac{Dd\tau''}{\sigma} \\ &\times\left[\cos(n, x)\frac{d}{dy'}\left(\frac{1}{\rho'}\right) - \cos(n, y)\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right] \\ &-\frac{d}{dz}\left(\frac{1}{\rho_k}\right)\int_i \frac{Dd\tau''}{\sigma}\left[\cos(n, z)\frac{d}{dx'}\left(\frac{1}{\rho'}\right) - \cos(n, x)\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\Big\} \\ C_k = \frac{1}{4\pi\mathfrak{V}^2}\int Md\tau'\Big\{&\frac{d}{dy}\left(\frac{1}{\rho_k}\right)\int Ld\tau''\left[X\frac{d}{dy'}\left(\frac{1}{\rho'}\right) - Y\frac{d}{dx'}\left(\frac{1}{\rho'}\right)\right] \\ &-\frac{d}{dz}\left(\frac{1}{\rho_k}\right)\int Ld\tau''\left[Z\frac{d}{dx'}\left(\frac{1}{\rho'}\right) - X\frac{d}{dz'}\left(\frac{1}{\rho'}\right)\right]\Big\} \end{aligned}\right\}, (25)$$

with similar expressions for the coefficients $A_{ik}', \ldots C_k''$; these coefficients are evidently entirely independent of the values of the quantities P, Q, R, α, β, γ, ϕ.

Lastly, to obtain a system of differential equations for the determination of the current-strengths I_k we form the expression (13) from these values (24) for P, Q, R, and we find

$$\int_k (P\,dx + Q\,dy + R\,dz)$$
$$= \int_k \left(\sum_{i=1}^{i=i} A_{ik} \frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik} \frac{d^2 I_i}{dt^2} + C_k \right) dx$$
$$+ \int_k \left(\sum_{i=1}^{i=i} A_{ik}' \frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik}' \frac{d^2 I_i}{dt^2} + C_k' \right) dy$$
$$+ \int_k \left(\sum_{i=1}^{i=i} A_{ik}'' \frac{dI_i}{dt} + \sum_{i=1}^{i=i} B_{ik}'' \frac{d^2 I_i}{dt^2} + C_k'' \right) dz = I_k \int_k \frac{ds}{\sigma L}, \quad \ldots (26)$$

where the integrations are extended round the circuit k. We observe that we have hereby eliminated the electrostatic potential ϕ. As this differential equation holds for every circuit k of the given system, we thus obtain a system of differential equations of the second order for the determination of the current-strengths $I_1, I_2, \ldots I_i$, which we can evidently write in the form

$$\left.\begin{array}{l}
I_1 = a_{11}\dfrac{dI_1}{dt} + a_{21}\dfrac{dI_2}{dt} + \ldots\, b_{11}\dfrac{d^2I_1}{dt^2} + b_{21}\dfrac{d^2I_2}{dt^2} + \ldots\, c_1 \\
I_2 = a_{12}\dfrac{dI_1}{dt} + a_{22}\dfrac{dI_2}{dt} + \ldots\, b_{12}\dfrac{d^2I_1}{dt^2} + b_{22}\dfrac{d^2I_2}{dt^2} + \ldots\, c_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
I_i = a_{1i}\dfrac{dI_1}{dt} + a_{2i}\dfrac{dI_2}{dt} + \ldots\, b_{1i}\dfrac{d^2I_1}{dt^2} + b_{2i}\dfrac{d^2I_2}{dt^2} + \ldots\, c_i
\end{array}\right\}, \quad \ldots (27)$$

where
$$a_{ik} = \frac{\displaystyle\int_k (A_{ik}\,dx + A_{ik}'\,dy + A_{ik}''\,dz)}{\displaystyle\int_k \frac{ds}{\sigma L}},$$

$$b_{ik} = \frac{\int\limits_k (B_{ik}dx + B_{ik}'dy + B_{ik}''dz)}{\int\limits_k \frac{ds}{\sigma L}},$$

and $$c_{ik} = \frac{\int\limits_k (C_k dx + C_k'dy + C_k''dz)}{\int\limits_k \frac{ds}{\sigma L}}.$$

We observe that these new coefficients a_{ik}, b_{ik}, and c_k are functions only of the above coefficients $A_{ik}A'_{ik}A''_{ik}$, $B_{ik}B'_{ik}B''_{ik}$, and $C_kC'_kC''_k$ respectively and of the configuration and conductivity (L) of the circuit k; as the last coefficients are functions of the external electromotive forces residing in the given system, they and hence the c_k's may generally be put equal to zero.

The complete solution of the above system (27) of differential equations would require a knowledge of the initial conditions at every point of the given system. The resulting solutions would be first approximations; second and higher approximations could then be found by starting from these new values or first approximations for P, Q, R, given by formulae (21), and by repeating the above operations. We may observe here supplementarily that it is customary to assume that the given electric currents behave like stationary electric currents in all sections of their circuits that cannot be regarded as approximately linear, since the exact determination of the distribution of the current-flow in such regions would necessitate approximations of a higher order than those just considered.

In order that our concrete representation may include the changes that formulae (25, XI.) have undergone upon the appearance of the quantities

$$\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}, \quad \overline{L(Q+Y)+\frac{D}{4\pi}\frac{dQ}{dt}},$$

and
$$\overline{L(R+Z)+\frac{D}{4\pi}\frac{dR}{dt}}$$

(cf. formulae (7)) in place of $\bar{p}, \bar{q}, \bar{r}$, we must assume that in addition to the galvanic current, whose components are

$$p=L(P+X), \quad q=L(Q+Y), \quad r=L(R+Z),$$

there is a second current, whose components are

$$\frac{D}{4\pi}\frac{dP}{dt}, \quad \frac{D}{4\pi}\frac{dQ}{dt}, \quad \frac{D}{4\pi}\frac{dR}{dt}, \qquad \text{(28)}$$

and that the respective components of these two currents are superposed in equations (7); hereby we must of course assume that the latter components have the same inductive effect as the former and hence, in conformity to the principle of the conservation of energy, the same magnetic and electrodynamic effect.

By formula (16, VI.) the electric moment (per unit-volume) along the x-axis due to electric polarization was

$$\mathfrak{x}=\frac{D-\mathfrak{k}}{4\pi}P.$$

We conceived that this moment $\mathfrak{x}$ was due to the appearance of the quantity

$$\frac{D-\mathfrak{k}}{4\pi}P\,dy\,dz$$

of negative electric fluid on the one $dy\,dz$ side of the volume-element $dx\,dy\,dz$ and to the displacement of the same quantity of positive electric fluid from that to its opposite side. Such a displacement evidently corresponds to an infinitely short current of current-strength or density

$$\frac{D-\mathfrak{k}}{4\pi}P\,dy\,dz \quad \text{or} \quad \frac{D-\mathfrak{k}}{4\pi}P$$

respectively. Any change in the electric polarization of the field would thus be accompanied by such an electric current of the component current-densities

$$\frac{D-\mathfrak{k}}{4\pi}\frac{dP}{dt},\quad \frac{D-\mathfrak{k}}{4\pi}\frac{dQ}{dt},\quad \frac{D-\mathfrak{k}}{4\pi}\frac{dR}{dt}. \qquad\qquad (29)$$

We shall call this current the electric polarization- or induction-current.

A comparison of the current-densities (29) with those (28) of the new current of our concrete representation shows that the former are $\frac{D-\mathfrak{k}}{4\pi}$ times as large as those of the latter. To obtain Maxwell's equations from our concrete representation we must therefore assume that the electrodynamic action of the electricity displaced by electric polarization (during its displacement), that is, that of the electric induction-current, is $\frac{D}{D-\mathfrak{k}}$ times as strong as that of the galvanic current itself; if we put the arbitrary constant $\mathfrak{k}$ equal to zero, this factor reduces to unity and the electrodynamic action of the induction-current thus becomes the same as that of the galvanic; in this case it is evidently immaterial whether we suppose that this fictitious electricity that acts electrodynamically flows in the galvanic current or is due to electric polarization.

The theory of action at a distance defined by our present concrete representation is evidently only a modified form of Maxwell's theory. Not only all problems in aphotic motion but all other disturbances of the ether, whether treated according to this modified form of Maxwell's theory or Maxwell's theory itself, lead to the same results, and on the other hand all experiments fail to offer any means of deciding in favour of either. Suppose, for example, that the positive and negative fluids of any given volume-element are suddenly separated; equal quantities of

electric fluid of opposite kind are then liberated in its adjacent volume-elements by electric polarization; the action at a distance of the free electric fluids of these volume-elements evidently counterbalances that of the fluids of the given volume-element, so that no action can manifest itself at any distant point, until the electric oscillations themselves have reached it; it is then that the action of the electric induction-current of our concrete representation makes its appearance at that point.

Lastly, we must assume that in our concrete representation the arbitrary constant $\mathfrak{e}$ is so small that the quantity $(D-\mathfrak{e})$ is positive for all bodies or media, since a negative value would evidently lead to a labil equilibrium. In media, for which $(D-\mathfrak{e})$ is infinitely small, the action of the electric induction-current at any distant point would of course be infinite. If we should assume that the electrodynamic action of the electric induction-current were exactly the same as that of the galvanic instead of $\frac{D}{D-\mathfrak{e}}$ times that of the latter, we should be able to determine the constant $\mathfrak{e}$ experimentally; such a determination would, however, only have a signification in our concrete representation.

In any medium, where $L=X=Y=Z=0$, formulae (6) and (7) evidently assume analogous forms, and the quantities D, P, Q, R, ϕ and $M, \alpha, \beta, \gamma, \psi$ respectively may thus be interchanged with one another. It follows therefore that magnetism behaves exactly like electricity in any non-conductor, within which no electromotive forces reside. Moreover, since formulae (7) are here analogous to those (25, XI.) for stationary flow, it follows that an analogy exists between magnetic polarization (induction) and stationary flow. This last analogy differs however from the first in the one respect that it cannot be extended to action at a distance and to induction, since stationary currents act on magnets but only magnets of variable intensity on real electricity—a magnetic field of constant intensity exercises, for example, no ponderable

force whatever on quantities of electricity placed within it. This is evidently true of our second analogy also. It follows moreover from the analogy between electric polarization and stationary electric currents that two paraffin rings, within which the electric induction (at right angles to their respective meridians) changes very rapidly, should attract or repel each other like two (respectively flowing) stationary currents. Such paraffin rings have, in fact, been investigated by Hertz* and have been found to possess this property. Finally it follows from the third or last of the above analogies, that, namely, between electricity and magnetism, that rapid variations in the magnetic induction of iron rings should give rise to ponderable forces between them. Hertz † has also called attention to this phenomenon.

Lastly, let us examine the characteristic form assumed by Maxwell's equations of action at a distance in a good annular ring or conductor, due to the presence of external electric currents or magnets, whose rates of change of current-strength and intensity respectively remain approximately constant during a given period. The state of the ether within such a conductor can evidently be regarded as approximately stationary during the given period, so that P, Q, R become here the derivatives of a given function, and hence the expression

$$Pdx + Qdy + Rdz$$

a complete differential. It follows therefore from formulae (6) that the expression

$$\frac{1}{4\pi\mathfrak{B}}\left\{dx\frac{d}{dt}\left(\frac{d\overline{M\gamma}}{dy} - \frac{d\overline{M\beta}}{dz}\right) + dy\frac{d}{dt}\left(\frac{d\overline{M\alpha}}{dz} - \frac{d\overline{M\gamma}}{dx}\right) + dz\frac{d}{dt}\left(\frac{d\overline{M\beta}}{dx} - \frac{d\overline{M\alpha}}{dy}\right)\right\}$$

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* "*Untersuchungen über die Ausbreitung der electrischen Kraft,*" p. 7.

† Widemann's *Annalen*, v. 23, p. 84, 1884.

is a complete differential of a given (double-valued) function χ. Hence we have

$$P=-\frac{d\phi}{dx}+\frac{d\chi}{dx},\quad Q=-\frac{d\phi}{dy}+\frac{d\chi}{dy},\quad R=-\frac{d\phi}{dz}+\frac{d\chi}{dz}.$$

If we put

$$P-\frac{d\chi}{dx}=-\frac{d\phi}{dx}=P_1,\quad Q-\frac{d\chi}{dy}=-\frac{d\phi}{dy}=Q_1,$$

$$R-\frac{d\chi}{dz}=-\frac{d\phi}{dz}=R_1,$$

P_1, Q_1, R_1 also become the partial derivatives of a given (single-valued) function with regard to the coordinates.

Since the function χ is independent of the time, we can then write equations (9, II.) as follows:

$$\frac{D}{\mathfrak{V}}\frac{dP_1}{dt}=\frac{d\beta}{dz}-\frac{d\gamma}{dy}-4\pi\frac{L}{\mathfrak{V}}\left(P_1+\frac{d\chi}{dx}\right), \quad \ldots\ldots\ldots(30)$$

with similar equations in Q_1 and R_1.

These equations in P_1, Q_1, R_1 are now identical with those that hold for P, Q, R, when we write

$$\frac{d\chi}{dx}=X,\quad \frac{d\chi}{dy}=Y,\quad \frac{d\chi}{dz}=Z,$$

where X, Y, Z denote the external electromotive forces acting in the system (P, Q, R). That the latter equations may be identical with the former in every respect, P, Q, R must likewise be the partial derivatives of a given function with regard to the coordinates; this condition evidently corresponds to a stationary (aphotic) state of the ether, that is, to the exclusion of all electric currents of variable current-strength and magnets of variable intensity from the given system. It follows therefore that the quantities P_1, Q_1, R_1 of the problem in induction will have the same values as the quantities P, Q, R of that in stationary flow. Since the component current-densities in the former problem, namely

$$L\left(P_1+\frac{d\chi}{dx}\right)=LP,\quad L\left(Q_1+\frac{d\chi}{dy}\right)=LQ,\quad L\left(R_1+\frac{d\chi}{dz}\right)=LR$$

and those in the latter, namely

$$L(P_1+X),\quad L(Q_1+Y),\quad L(R_1+Z)$$

—we have written here P_1, Q_1, R_1 instead of P, Q, R, since the P, Q, R of this problem have the same values as the P_1, Q_1, R_1 of the other—must evidently be equal to each other, it follows that the induced currents of the former must obey the same laws as those, to which the external electromotive forces of the latter would give rise. Lastly, we observe that the component current-densities of these induced currents are proportional to the partial derivatives of the (double-valued) function χ with regard to the coordinates.

We have now already seen in § 19 that the solution of the above problem in stationary flow is given by the function

$$\phi=\chi-k\chi',$$

where χ' is a function only of the configuration of the given conductor and k the potential difference between the terminals, and, moreover, that the current-flow is proportional to this constant k and entirely independent of the form of the function χ (cf. p. 161) (χ is here identical to ψ of § 19); it follows therefore from the above analogy between the two given problems—this analogy forms a new feature of our concrete representation—that the flow of induced currents in the given annular conductor must be the same as the stationary flow of electricity through it, provided only the constant k has the same value in both problems.

Lastly, we saw in § 23 that, when a good conductor is replaced by a dielectric, whose constant of electric polarization is large in comparison to that of the surrounding medium, the equations that defined the stationary state of the ether in the latter were similar

to those that hold for the former, differing from them only in the appearance of the constant D in place of the constant $4\pi L$. Since both the components of the electric moment and those of the current-density are proportional to the quantities P, Q, R, it thus follows that the electric polarization in such a ring, due to the presence of electric currents (magnets) whose rates of change of current-strength remain approximately constant, must behave similarly to the current-density in the corresponding conductor, and from the further analogy between electricity and magnetism on p. 192 that the behaviour of the magnetic polarization in a ring, whose constant of magnetic polarization M is large in comparison to that of the surrounding medium, due to slow changes in the magnetic induction of the field, must be similar to that of the electric current in the analogous conductor.

CHAPTER XVI.

SECTION XXXVI. MODIFIED FORMS OF MAXWELL'S EQUATIONS OF ACTION AT A DISTANCE.

THE equations of the preceding chapter should be regarded as only a special form of Maxwell's equations of action at a distance, obtained by putting the arbitrary constant $\mathfrak{m}$ of formula (5, XI.) equal to zero. Let us next consider a somewhat more general form. As the constant $\mathfrak{m}$ must be given the same value in all bodies or media and that value must be retained through all periods, our former equations will evidently undergo changes of only a purely formal character by its introduction. To obtain the desired form most readily, we replace in our fundamental equations (10, II.) the medium-constant M by the quantity $(M'+\mathfrak{m})$, differentiate partially, the second equation with regard to z and the third to y, and subtract; we find then

$$\frac{\mathfrak{m}}{\mathfrak{V}}\frac{d}{dt}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\frac{1}{\mathfrak{V}}\frac{d}{dt}\left[\frac{d(M'\beta)}{dz}-\frac{d(M'\gamma)}{dy}\right]$$
$$=\nabla^2 P-\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)$$

and similar differential equations in Q and R.

This equation evidently gives

$$P=-\frac{1}{4\pi}\int\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\frac{d\tau'}{\rho}-\frac{\mathfrak{m}}{4\pi\mathfrak{V}}\int\frac{d}{dt}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)\frac{d\tau'}{\rho}$$
$$+\frac{1}{4\pi\mathfrak{V}}\int\frac{d}{dt}\left[\frac{d(M'\gamma)}{dy}-\frac{d(M'\beta)}{dz}\right]\frac{d\tau'}{\rho}.$$

Replacing here $\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)$ by its value from equations (9, II.) and M' by $(M-\mathfrak{m})$ we have

$$P=-\frac{d\phi}{dx}-\frac{\mathfrak{m}}{\mathfrak{B}^2}\frac{d}{dt}\left[\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}\right]$$
$$+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\gamma}}{dy}-\frac{\overline{d(M-\mathfrak{m})\beta}}{dz}\right], \quad \text{......(1)}$$

where the horizontal dashes are used as symbols and have the same meaning as in the preceding articles; similar expressions hold for Q and R. These three formulae replace formulae (6, XV.); on the other hand formulae (7, XV.) undergo no changes whatever from the introduction of the arbitrary constant $\mathfrak{m}$. These six equations together with the two conditional relations (9, XV.) and (10, XV.) and the condition that both ϕ and ψ vanish at infinite distance, replace the corresponding equations of the preceding chapter in every respect. We observe that the component electric forces P, Q, R are represented here as the sum of three component electric forces; the first terms of these expressions represent the electrostatic component forces; we thus know from our concrete representation that the electrostatic force acting at any point is always equal to that exerted on unit gravitating mass, placed at that point, by a gravitating mass, whose density at every point of space is given by the expression

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right);$$

wherever this expression is negative, we must of course conceive that the density of this fictitious mass is there negative, that is, that it repels instead of attracts the given (unit) mass (cf. § 13.)

The second terms of the above expressions for P, Q, R represent the components of the electrodynamic force. To obtain an illustrative interpretation for this force, we

recall the new feature of our concrete representation introduced in the preceding article, according to which, in addition to the galvanic current with the component current-densities

$$p_1 = L(P+X), \quad q_1 = L(Q+Y), \quad r_1 = L(R+Z),$$

there was a second current, represented by the electric induction-current, whose densities were

$$p_2 = \frac{D}{4\pi}\frac{dP}{dt}, \quad q_2 = \frac{D}{4\pi}\frac{dQ}{dt}, \quad r_2 = \frac{D}{4\pi}\frac{dR}{dt}.$$

The total current-densities would thus be

$$p = p_1 + p_2, \quad q = q_1 + q_2, \quad r = r_1 + r_2.$$

The electrodynamic potential of two linear elements ds and ds' of current-strengths i and i' is now given by the expression

$$\frac{Mii'dsds'\cos(ds, ds')}{\mathfrak{V}^2\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(cf. formula (10, XIII.)), whereas the component along any direction of the ponderable force exercised on any current-element ids is evidently the derivative with regard to that direction of the electrodynamic potential between all the other current-elements of the field and the given current-element (cf. pp. 313-315). To find the electrodynamic potential between the current in any volume-element $dx'dy'dz'$ of our given system and any given element ds of current-strength i, we observe that the component current-strengths in the given element are $pdy'dz'$, $qdx'dz'$ and $rdx'dy'$; the electrodynamic potential between the component current-strength $pdy'dz'$ and a given linear element dx of current-strength i is thus

$$\frac{Mipdy'dz'dxdx'\cos(dx, dx')}{\mathfrak{V}^2\rho} = \frac{Mipdxd\tau'}{\mathfrak{V}^2\rho};$$

and hence the total electrodynamic potential between the

component of the total current-strength along the x-axis and a linear element $dx=1$ of unit current-strength

$$M\int\frac{p\,d\tau'}{\mathfrak{B}^2\rho}=\frac{M}{\mathfrak{B}^2}\int\frac{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}{\rho}d\tau'$$

$$=\frac{M}{\mathfrak{B}^2}\left[\overline{L(P+X)+\frac{D}{4\pi}\frac{dP}{dt}}\right]$$

—we have assumed here that M is constant. It thus follows that the second term of the above expression (1) for P is $\frac{\mathfrak{m}}{M}$ times the negative derivative with regard to x of the electrodynamic potential between all the electric currents of the field and unit linear element of unit current-strength parallel to the x-axis; hereby we have found the desired illustrative interpretation for the second term of the above expression for P—the respective terms of the given expressions for Q and R can be similarly interpreted.

Lastly, to interpret the third terms of the expressions (1) for P, Q, R, those due to variations in the rate of change in the magnetic polarization of the field, it is not necessary that we should have recourse to any new conceptions, but only that we should assume, in the first place, that the medium contains small electric circuits (molecular currents), which arrange themselves in some predominating plane (cf. text, p. 256) when the given field is generated, or, provided such currents do not already exist, that they are created in the given plane upon the production of the field, and, secondly, that these currents thereby overcome molecular forces that are proportional to the components α, β, γ of the magnetic polarization, such, in fact, that their component current-densities are given by the quantities

$$\frac{M-\mathfrak{m}}{4\pi}\alpha,\quad\frac{M-\mathfrak{m}}{4\pi}\beta,\quad\frac{M-\mathfrak{m}}{4\pi}\gamma,$$

where α, β, γ denote the components of the force exercised by all the currents of the field, including its molecular currents of constant current-strength (permanent magnets), on the end of a solenoid, whose current-strength i is given by the relation

$$\frac{ifMN}{\mathfrak{B}} = 1.$$

In choosing this definition for α, β, γ, that is, in designating these quantities as the components of the magnetic force or field-strength, we are also obtaining an interpretation for formulae (7, XV.). Lastly, upon assuming that the electrodynamic action of these molecular currents is the same as that of electric currents proper, we can interpret the third terms of the expressions for P, Q, R, for they may then be designated as the components of the electrodynamic induction (force).

Finally, to interpret the conditional relation (9, XV.)—the other condition (10, XV.) demands the exclusion of all real magnetism from the field—we have recourse to the conceptions of the flow of the electric fluids (neutral electricities), § 6, and of their displacement (polarization), § 14.

Conversely, the above conceptions, features of our concrete representation, introduced here for interpreting equations (6, 7, 9 and 10, XV.) could be taken as fundament of our theory of electricity and magnetism—this is, in fact, done in the old theory of action at a distance—and the given equations deduced from them. For example, condition (10, XV.) would follow directly from the supposition that magnetism is the magnetic effect created by molecular currents in certain bodies, as iron, and from the natural assumption that the lines of magnetic induction are always closed. After having thus established equations (6, 7, 9 and 10, XV.), we could then derive Maxwell's equations (9, II.) and (10, II.), which form the fundament of the present theory, from the former by the elimination of the functions ϕ and ψ.

A second purely formal but important change can be made in Maxwell's equations by introducing the external electromotive forces X, Y, Z in quite another form, namely, by putting

$$LX = L\mathfrak{X} + \frac{D-\mathfrak{e}}{4\pi}\frac{d\mathfrak{X}}{dx}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

with similar formulae for Y and Z, and by assuming that $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ were initially, $t = -\infty$, zero. These differential equations and initial conditions evidently determine $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ uniquely. The integrals of these equations can be found by formula (14, VIII.); we have, namely,

$$\mathfrak{X} = \frac{4\pi L}{D-\mathfrak{e}} e^{\frac{-4\pi Lt}{D-\mathfrak{e}}} \int_{-\infty}^{t} X e^{\frac{4\pi Lt}{D-\mathfrak{e}}} dt$$

and similar expressions for $\mathfrak{Y}$ and $\mathfrak{Z}$; to evaluate these integrals, X, Y, Z must, of course, be given as functions of the time.

If X, Y, Z are constant during any period, we have

$$\mathfrak{X} = X, \quad \mathfrak{Y} = Y, \quad \mathfrak{Z} = Z.$$

As these relations correspond to the only state of the ether that has been more carefully investigated experimentally, it is impossible to decide whether we should regard the quantities X, Y, Z or $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ as our external electromotive forces.

By the substitution of the above values (3) for X, Y, Z in equations (9, II.) the latter assume the form

$$\frac{D}{\mathfrak{B}}\frac{dP}{dt} + \frac{D-\mathfrak{e}}{\mathfrak{B}}\frac{d\mathfrak{X}}{dt} = \frac{d\beta}{dz} - \frac{d\gamma}{dy} - \frac{4\pi L}{\mathfrak{B}}(P + \mathfrak{X}), \ldots\ldots (4)$$

with similar equations in Q and R. Equations (10, II.) remain unchanged. Our equations of action at a distance (1) and (7, XV.) then become

$$P=-\frac{d\phi}{dx}-\frac{\mathfrak{m}}{\mathfrak{V}^2}\frac{d}{dt}\left[\overline{L(P+\mathfrak{X})+\frac{D}{4\pi}\frac{dP}{dt}+\frac{D-\mathfrak{e}}{4\pi}\frac{d\mathfrak{X}}{dt}}\right]$$

$$+\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\gamma}}{dy}-\frac{\overline{d(M-\mathfrak{m})\beta}}{dz}\right]\text{.............}(5)$$

and

$$\alpha=-\frac{d\psi}{dx}+\frac{1}{\mathfrak{V}}\left\{\frac{d}{dz}\left[\overline{L(Q+\mathfrak{Y})+\frac{D}{4\pi}\frac{dQ}{dt}+\frac{D-\mathfrak{e}}{4\pi}\frac{d\mathfrak{Y}}{dt}}\right]\right.$$

$$\left.-\frac{d}{dy}\left[\overline{L(R+\mathfrak{Z})+\frac{D}{4\pi}\frac{dR}{dt}+\frac{D-\mathfrak{e}}{4\pi}\frac{d\mathfrak{Z}}{dt}}\right]\right\},\text{ ....}(6)$$

with similar expressions for P, Q and β, γ respectively. Equation (9, XV.) assumes the form

$$\frac{1}{4\pi}\frac{d}{dt}\left\{\frac{d}{dx}[D(P+\mathfrak{X})-\mathfrak{e}\mathfrak{X}]+\frac{d}{dy}[D(Q+\mathfrak{Y})-\mathfrak{e}\mathfrak{Y}]\right.$$

$$\left.+\frac{d}{dz}[D(R+\mathfrak{Z})-\mathfrak{e}\mathfrak{Z}]\right\}+\frac{d}{dx}L(P+\mathfrak{X})$$

$$+\frac{d}{dy}L(Q+\mathfrak{Y})+\frac{d}{dz}L(R+\mathfrak{Z})=0$$

or

$$\frac{1}{4\pi}\frac{d}{dt}\left[\frac{d}{dx}(D-\mathfrak{e})(P+\mathfrak{X})+\frac{d}{dy}(D-\mathfrak{e})(Q+\mathfrak{Y})\right.$$

$$\left.+\frac{d}{dz}(D-\mathfrak{e})(R+\mathfrak{Z})\right]+\frac{d}{dx}L(P+\mathfrak{X})+\frac{d}{dy}L(Q+\mathfrak{Y})$$

$$+\frac{d}{dz}L(R+\mathfrak{Z})=-\frac{\mathfrak{e}}{4\pi}\frac{d}{dt}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)=-\frac{d\epsilon_f}{dt}\text{....}(7)$$

It follows from equation (5) that for $\mathfrak{e}=0$ the external electromotive forces $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ would be equally distributed between the quantities

$$LP\text{ and }\frac{D}{4\pi}\frac{dP}{dt},\quad LQ\text{ and }\frac{D}{4\pi}\frac{dQ}{dt},\quad LR\text{ and }\frac{D}{4\pi}\frac{dR}{dt},$$

that is, between the first and second types of electricity, the so-called current and polarized electricities respectively of our concrete representation (cf. § 14). As the external electromotive forces act on the polarized electricity, we would thus have to assume henceforth the following values for the components of the electric moment unit (per volume) instead of the former ones (16, VI.):

$$\mathfrak{x}'=\frac{D-\mathfrak{k}}{4\pi}(P+\mathfrak{X}),\quad \mathfrak{y}'=\frac{D-\mathfrak{k}}{4\pi}(Q+\mathfrak{Y}),\quad \mathfrak{z}'=\frac{D-\mathfrak{k}}{4\pi}(R+\mathfrak{Z}),$$

which give

$$\left.\begin{aligned} P&=\frac{\mathfrak{x}'}{\epsilon}-\mathfrak{X},\quad Q=\frac{\mathfrak{y}'}{\epsilon}-\mathfrak{Y},\quad R=\frac{\mathfrak{z}'}{\epsilon}-\mathfrak{Z},\\ \text{where}\qquad \epsilon&=\frac{D-\mathfrak{k}}{4\pi}\end{aligned}\right\}\quad\ldots\ldots\ldots(8)$$

The density of the free electricity ϵ_f' must then be written

$$\begin{aligned}\epsilon_f'&=\frac{\mathfrak{k}}{4\pi}\left[\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right]\\ &=\frac{\mathfrak{k}}{4\pi}\left[\frac{d}{dx}\left(\frac{\mathfrak{x}'}{\epsilon}-\mathfrak{X}\right)+\frac{d}{dy}\left(\frac{\mathfrak{y}'}{\epsilon}-\mathfrak{Y}\right)+\frac{d}{dz}\left(\frac{\mathfrak{z}'}{\epsilon}-\mathfrak{Z}\right)\right];\ldots.(9)\end{aligned}$$

equation (7) thus becomes

$$\frac{d\epsilon_f'}{dt}+\frac{d}{dt}\left[\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right]+\frac{d}{dx}\left(\frac{L\mathfrak{x}'}{\epsilon}\right)+\frac{d}{dy}\left(\frac{L\mathfrak{y}'}{\epsilon}\right)+\frac{d}{dz}\left(\frac{L\mathfrak{z}'}{\epsilon}\right)=0$$

or, if we put

$$u'=\frac{d\mathfrak{x}'}{dt}+\frac{\mathfrak{x}'}{\kappa\epsilon},\quad v'=\frac{d\mathfrak{y}'}{dt}+\frac{\mathfrak{y}'}{\kappa\epsilon},\quad w'=\frac{d\mathfrak{z}'}{dt}+\frac{\mathfrak{z}'}{\kappa\epsilon},\quad \kappa=\frac{1}{L},\ldots.(10)$$

$$\frac{d\epsilon_f'}{dt}+\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=0.\ \ldots\ldots\ldots\ldots\ldots(11)$$

These equations (4)-(11) are identical with Maxwell's in every respect, they are, in fact, only another form of

the latter. The concrete representation that replaces them differs, however, materially not only from Maxwell's, but from the above; nevertheless, it must be possible to reproduce Maxwell's equations from it. Although this new concrete representation evidently depends on the value assigned the arbitrary constant $\mathfrak{e}$, the phenomena expressed by it and its equations are entirely independent of that value. These new equations (4)-(11) lead, however, to a more general theory of electricity and magnetism, known as von Helmholtz's theory, which is only identical with Maxwell's when $\mathfrak{e}$ is put equal to zero, whereas for $\mathfrak{e} \lessgtr 0$ all ensuing phenomena depend on the value assigned this constant.

SECTION XXXVII. TRANSITION FROM MAXWELL'S TO VON HELMHOLTZ'S EQUATIONS OF ACTION AT A DISTANCE.

To derive von Helmholtz's equations of action at a distance from Maxwell's we have only to modify the latter in the manner suggested by our concrete representation. According to the conceptions on p. 334 any variation in the electric polarization of our medium must give rise to a current, the so-called electric polarization current, whose component current-densities are

$$\epsilon\frac{d}{dt}(P+\mathfrak{X}),\quad \epsilon\frac{d}{dt}(Q+\mathfrak{Y}),\quad \epsilon\frac{d}{dt}(R+\mathfrak{Z}),\ldots\ldots(12)$$

whereas the component-densities of the current that must be added to those of the galvanic current in our new concrete representation are

$$\left.\begin{array}{c}\dfrac{D}{4\pi}\dfrac{dP}{dt}+\dfrac{D-\mathfrak{e}}{4\pi}\dfrac{d\mathfrak{X}}{dt},\quad \dfrac{D}{4\pi}\dfrac{dQ}{dt}+\dfrac{D-\mathfrak{e}}{4\pi}\dfrac{d\mathfrak{Y}}{dt}\\ \dfrac{D}{4\pi}\dfrac{dR}{dt}+\dfrac{D-\mathfrak{e}}{4\pi}\dfrac{d\mathfrak{Z}}{dt}\end{array}\right\}\ldots\ldots(13)$$

(cf. formulae (5)). A comparison of these two component-densities with each other suggests the introduction of the former in place of the latter in equations (4)-(11). It is this slight modification in Maxwell's equations that leads to von Helmholtz's equations of action at a distance. Equations (5) and (6) must then be written as follows:

$$P=-\frac{d\phi}{dx}-\frac{\mathfrak{m}}{\mathfrak{V}^2}\frac{d}{dt}\left[\overline{L(P+\mathfrak{X})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(P+\mathfrak{X})}\right]$$
$$+\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\gamma}}{dy}-\frac{\overline{d(M-\mathfrak{m})\beta}}{dz}\right]$$

and

$$a=-\frac{d\psi}{dx}+\frac{1}{\mathfrak{V}}\left\{\frac{d}{dz}\left[\overline{L(Q+\mathfrak{Y})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(Q+\mathfrak{Y})}\right]\right.$$
$$\left.-\frac{d}{dy}\left[\overline{L(R+\mathfrak{Z})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(R+\mathfrak{Z})}\right]\right\}$$

or, by formulae (8) and (11),

$$P=-\frac{d\phi}{dx}-\frac{\mathfrak{m}}{\mathfrak{V}^2}\frac{d\bar{u}'}{dt}+\frac{1}{4\pi\mathfrak{V}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\gamma}}{dy}-\frac{\overline{d(M-\mathfrak{m})\beta}}{dz}\right] \quad (14)$$

and
$$a=-\frac{d\psi}{dx}+\frac{1}{\mathfrak{V}}\left(\frac{d\bar{v}'}{dz}-\frac{d\bar{w}'}{dy}\right), \quad \ldots\ldots\ldots\ldots\ldots(15)$$

with similar expressions for Q, R and β, γ respectively.

We observe, however, that

$$\frac{d}{dx}\left[\overline{L(P+\mathfrak{X})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(P+\mathfrak{X})}\right]$$
$$+\frac{d}{dy}\left[\overline{L(Q+\mathfrak{Y})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(Q+\mathfrak{Y})}\right]$$
$$+\frac{d}{dz}\left[\overline{L(R+\mathfrak{Z})+\frac{D-\mathfrak{e}}{4\pi}\frac{d}{dt}(R+\mathfrak{Z})}\right]$$
$$=\frac{d\bar{u}'}{dx}+\frac{d\bar{v}'}{dy}+\frac{d\bar{w}'}{dz}=-\frac{d\bar{\epsilon}_f'}{dt} \quad \ldots\ldots\ldots\ldots\ldots(16)$$

(cf. formula (11)) and that formulae (14) give

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\gtrless\nabla^2\phi.$$

We must thus abandon our previous definition of ϕ as the potential of a mass, whose density at every point is given by the expression

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right),$$

and in its stead define ϕ as the potential of a mass whose density is $\epsilon_f'/\mathfrak{e}$, where ϵ_f' is given by formula (16), and modify the above equations accordingly. For this purpose we replace u', v', w' by

$$\bar{u}'+\frac{1-k}{2}\frac{d\chi}{dx},\quad \bar{v}'+\frac{1-k}{2}\frac{d\chi}{dy}\text{ and }\bar{w}'+\frac{1-k}{2}\frac{d\chi}{dz}$$

respectively, where k is an arbitrary constant and χ any function of u', v', w' that satisfies the condition

$$\frac{d}{dx}\left(\bar{u}'+\frac{1-k}{2}\frac{d\chi}{dx}\right)+\frac{d}{dy}\left(\bar{v}'+\frac{1-k}{2}\frac{d\chi}{dy}\right)$$
$$+\frac{d}{dz}\left(\bar{w}'+\frac{1-k}{2}\frac{d\chi}{dz}\right)=m\nabla^2\chi,\text{ ....(17)}$$

where m is a constant whose value is to be determined. Such a function is now

$$\chi=\int d\tau\left(u'\frac{d\rho}{d\xi}+v'\frac{d\rho}{d\eta}+w'\frac{d\rho}{d\zeta}\right),\text{ ....(18)}$$

where ξ, η, ζ denote the coordinates of any volume-element $d\tau$ and ρ the distance of the latter from the point at which the value of χ is sought. Integrate the first term of this integral-expression partially with regard to ξ and we get

$$\int d\tau u'\frac{d\rho}{d\xi}=\iint d\eta\, d\zeta\left|u'\rho\right|_{\xi=-\infty}^{\xi=\infty}-\int d\tau\frac{du'}{d\xi}\rho;$$

the first integral on the right evidently vanishes, and we have

$$\int d\tau u' \frac{d\rho}{d\xi} = -\int d\tau \rho \frac{du'}{d\xi}$$

and, similarly,

$$\int d\tau v' \frac{d\rho}{d\eta} = -\int d\tau \rho \frac{dv'}{d\eta} \text{ and } \int d\tau w' \frac{d\rho}{d\zeta} = -\int d\tau \rho \frac{dw'}{d\zeta}.$$

The given function χ can thus be written

$$\chi = -\int d\tau \rho \left(\frac{du'}{d\xi} + \frac{dv'}{d\eta} + \frac{dw'}{d\zeta} \right)$$

or, by formula (11),

$$\chi = \int d\tau \rho \frac{d\epsilon_f'}{dt} = \frac{d}{dt} \int d\tau \rho \epsilon_f' ; \quad \text{.................(19)}$$

from which value it follows that

$$\frac{d^2\chi}{dx^2} = \frac{d}{dt} \int d\tau \epsilon_f' \frac{d^2\rho}{dx^2} = \frac{d}{dt} \int d\tau \epsilon_f' \left(\frac{1}{\rho} - \frac{(x-\xi)^2}{\rho^3} \right)$$

and, similarly,

$$\frac{d^2\chi}{dy^2} = \frac{d}{dt} \int d\tau \epsilon_f' \left(\frac{1}{\rho} - \frac{(y-\eta)^2}{\rho^3} \right) \text{ and } \frac{d^2\chi}{dz^2} = \frac{d}{dt} \int d\tau \epsilon_f' \left(\frac{1}{\rho} - \frac{(z-\zeta)^2}{\rho^3} \right).$$

These relations give

$$\nabla^2 \chi = 2 \frac{d}{dt} \int \frac{\epsilon_f' d\tau}{\rho} = 2 \frac{d\bar{\epsilon}_f'}{dt} \quad \text{..................(20)}$$

or, since

$$\phi = \frac{1}{\mathfrak{k}} \int \frac{\epsilon_f' d\tau}{\rho} = \frac{\bar{\epsilon}_f'}{\mathfrak{k}},$$

$$\nabla^2 \chi = 2\mathfrak{k} \frac{d\phi}{dt}. \quad \text{..............................(21)}$$

Formulae (16), (17) and (20) give the following equation for the determination of the constant m:

$$-k\frac{d\bar{\epsilon}_f'}{dt}=2m\frac{d\bar{\epsilon}_f'}{dt},$$

hence

$$m=-\frac{k}{2}. \quad\text{......................(22)}$$

The conditional equation (17) can thus be written in the following forms:

$$\frac{d\bar{u}'}{dx}+\frac{d\bar{v}'}{dy}+\frac{d\bar{w}'}{dz}=-\tfrac{1}{2}\nabla^2\chi=-\frac{d\bar{\epsilon}_f'}{dt}=-\mathfrak{e}\frac{d\phi}{dt} \quad\text{......(23)}$$

and equations (14) and (15) as follows:

$$\left.\begin{aligned}
P&=-\frac{d\phi}{dx}-\frac{\mathfrak{m}}{\mathfrak{B}^2}\frac{d}{dt}\left(\bar{u}'+\frac{1-k}{2}\frac{d\chi}{dx}\right)\\
&\quad+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\gamma}}{dy}-\frac{\overline{d(M-\mathfrak{m})\beta}}{dz}\right]\\
Q&=-\frac{d\phi}{dy}-\frac{\mathfrak{m}}{\mathfrak{B}^2}\frac{d}{dt}\left(\bar{v}'+\frac{1-k}{2}\frac{d\chi}{dy}\right)\\
&\quad+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\alpha}}{dz}-\frac{\overline{d(M-\mathfrak{m})\gamma}}{dx}\right]\\
R&=-\frac{d\phi}{dz}-\frac{\mathfrak{m}}{\mathfrak{B}^2}\frac{d}{dt}\left(\bar{w}'+\frac{1-k}{2}\frac{d\chi}{dz}\right)\\
&\quad+\frac{1}{4\pi\mathfrak{B}}\frac{d}{dt}\left[\frac{\overline{d(M-\mathfrak{m})\beta}}{dx}-\frac{\overline{d(M-\mathfrak{m})\alpha}}{dy}\right]
\end{aligned}\right\}\text{... (24)}$$

and

$$\left.\begin{aligned}
\alpha=-\frac{d\psi}{dx}+\frac{1}{\mathfrak{B}}\left(\frac{d\bar{v}'}{dz}-\frac{d\bar{w}'}{dy}\right),\quad \beta=-\frac{d\psi}{dy}+\frac{1}{\mathfrak{B}}\left(\frac{d\bar{w}'}{dx}-\frac{d\bar{u}'}{dz}\right)\\
\gamma=-\frac{d\psi}{dz}+\frac{1}{\mathfrak{B}}\left(\frac{d\bar{u}'}{dy}-\frac{d\bar{v}'}{dx}\right)
\end{aligned}\right\}.\text{(25)}$$

This is evidently the desired or modified form of equations (14) and (15), since they give

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=-\nabla^2\phi-\frac{\mathfrak{m}}{\mathfrak{B}^2}\frac{d}{dt}\left(\frac{d\bar{u}'}{dx}+\frac{d\bar{v}'}{dy}+\frac{d\bar{w}'}{dz}+\frac{1-k}{2}\nabla^2\chi\right),$$

which by formula (23) can be written

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=-\nabla^2\phi+\frac{k\mathfrak{m}}{2\mathfrak{V}^2}\frac{d}{dt}\nabla^2\chi=\nabla^2\left(\frac{k\mathfrak{m}}{2\mathfrak{V}^2}\frac{d\chi}{dt}-\phi\right);$$

that is, the expression

$$\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)$$

can again be regarded as the density of a mass, whose potential at every point is given by the function

$$\phi-\frac{k\mathfrak{m}}{2\mathfrak{V}^2}\frac{d\chi}{dt}.$$

The six equations (24) and (25), together with the two conditional equations (11) and (10, XV.), are the desired modified form of Maxwell's equations of action at a distance. Since ϕ is the potential of a mass of the density $\frac{\epsilon_f'}{\mathfrak{k}}$, the former of these conditions can also be written

$$-\frac{d\epsilon_f'}{dt}=\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=\frac{\mathfrak{k}}{4\pi}\frac{d}{dt}\nabla^2\phi. \quad \text{........(26)}$$

In von Helmholtz's notation, where

$$\left.\begin{array}{c}
\frac{1}{\mathfrak{V}}=A \quad \mathfrak{m}=1 \quad M=1+4\pi\theta, \\
\alpha=\frac{\lambda}{\theta} \quad \beta=\frac{\mu}{\theta} \quad \gamma=\frac{\nu}{\theta}, \\
\frac{M-1}{4\pi}\bar{\alpha}=[\theta\bar{\alpha}]=L,\ \frac{M-1}{4\pi}\bar{\beta}=[\theta\bar{\beta}]=M, \\
\frac{M-1}{4\pi}\bar{\gamma}=[\theta\bar{\gamma}]=N, \\
\bar{u}'+\frac{1-k}{2}\frac{d\chi}{dx}=U',\ \bar{v}'+\frac{1-k}{2}\frac{d\chi}{dy}=V',\ \bar{w}'+\frac{1-k}{2}\frac{d\chi}{dz}=W', \\
\psi=\chi \text{ and } \epsilon_f'=E,
\end{array}\right\} (27)$$

the above system of equations assumes the following form:

$$\left.\begin{aligned}\frac{\mathfrak{x}'}{\epsilon}&=-\frac{d\phi}{dx}-A^2\frac{dU'}{dt}+A\frac{d}{dt}\left(\frac{dN}{dy}-\frac{dM}{dz}\right)+\mathfrak{X}\\ \frac{\mathfrak{y}'}{\epsilon}&=-\frac{d\phi}{dy}-A^2\frac{dV'}{dt}+A\frac{d}{dt}\left(\frac{dL}{dz}-\frac{dN}{dx}\right)+\mathfrak{Y}\\ \frac{\mathfrak{z}'}{\epsilon}&=-\frac{d\phi}{dz}-A^2\frac{dW'}{dt}+A\frac{d}{dt}\left(\frac{dM}{dx}-\frac{dL}{dy}\right)+\mathfrak{Z},\end{aligned}\right\}\quad\ldots(28)$$

$$\left.\begin{aligned}\frac{\lambda}{\theta}&=-\frac{d\chi}{dx}+A\left(\frac{dV'}{dz}-\frac{dW'}{dy}\right)\\ \frac{\mu}{\theta}&=-\frac{d\chi}{dy}+A\left(\frac{dW'}{dx}-\frac{dU'}{dz}\right)\\ \frac{\nu}{\theta}&=-\frac{d\chi}{dz}+A\left(\frac{dU'}{dy}-\frac{dV'}{dx}\right),\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(29)$$

$$-\frac{dE}{dt}=\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=\frac{\epsilon}{4\pi}\frac{d}{dt}\nabla^2\phi \quad\ldots\ldots.(30)$$

and $$\frac{d}{dx}\left(\frac{\lambda}{\theta}\right)+\frac{d}{dy}\left(\frac{\mu}{\theta}\right)+\frac{d}{dz}\left(\frac{\nu}{\theta}\right)+4\pi\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)=0$$

or, by equations (29), which give

$$\frac{d}{dx}\left(\frac{\lambda}{\theta}\right)+\frac{d}{dy}\left(\frac{\mu}{\theta}\right)+\frac{d}{dz}\left(\frac{\nu}{\theta}\right)=-\nabla^2\chi,$$

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{1}{4\pi}\nabla^2\chi, \quad\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

hence

$$\chi=-\int\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\frac{d\tau}{\rho}.\quad\ldots\ldots\ldots\ldots\ldots(32)$$

The above equations (28)-(31) are identical with those developed by von Helmholtz* in every respect, except

* "*Ueber die Bewegungsgleichungen der Electricität für ruhende leitende Körper*"; equations (19*e*), (19*f*), (19*g*) and (20*f*) respectively.

that he puts $\mathfrak{e}=1$, whereas we have left this constant entirely arbitrary; that is, we have

$$\epsilon_{vH}=\frac{D-1}{4\pi}$$

and hence $$D=1+4\pi\epsilon_{vH}.$$

We have seen in § 17 that von Helmholtz's ideal standard medium is characterized by the relation $D_i=\mathfrak{e}_i$, where $\mathfrak{e}_i$ is supposed to be very small in comparison to unity, and that the constant $\mathfrak{e}_{vH}$ for any other body or medium referred to this medium is

$$\epsilon_{vH}=\frac{D-1}{4\pi},$$

where D denotes the inductive capacity of the given body measured in the ideal standard medium (cf. p. 131); that is, von Helmholtz measures all quantities in the electrostatic system of units of his ideal standard medium, whereas we employ the electrostatic system of our real standard medium, air. Our arbitrary constant $\mathfrak{e}$ could thus be defined as the inductive capacity of the ideal standard medium in units of the real standard medium (cf. also p. 132).

Observe that for $\mathfrak{e}=0$ von Helmholtz's equations of action at a distance reduce to Maxwell's, whatever value we assign the constant k; this point of similarity between von Helmholtz's and Maxwell's theories was first called attention to by Poincaré.* For $\mathfrak{e}\gtrless 0$ but $k=0$, we evidently have

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=\nabla^2\phi$$

—see the formula for this expression on p. 354—(cf. p. 351); that is, ϕ may again be regarded as the

* "*Electricité et Optique*," part II., p. 112.

potential of a mass, whose density $\frac{\epsilon_f'}{\mathfrak{e}}$ at every point is given by the expression

$$\frac{\epsilon_f'}{\mathfrak{e}}=\frac{1}{4\pi}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right),$$

and equation (9) still remains valid in spite of the above generalization. In this case, however, von Helmholtz's equations do not reduce to Maxwell's as might be supposed.

CHAPTER XVII.

SECTION XXXVIII. DERIVATION OF VON HELMHOLTZ'S EQUATIONS OF ACTION AT A DISTANCE FROM EMPIRICAL LAWS.

We have seen in the preceding chapter how von Helmholtz's equations of electricity and magnetism can be obtained from Maxwell's by making certain changes in the latter. Let us now pursue von Helmholtz's own method and derive his equations directly from empirical laws. We start from Ohm's law, which can be written in the familiar form

$$E = JW.$$

If electromotive forces reside in every element ds of the given circuit, as in those of induced currents, we can write this law in the more general form

$$\int p\,ds = \int \kappa i\,ds, \qquad \text{.......................(1)}$$

where p denotes the electromotive force acting in the element ds, i the current-strength and κ the specific resistance of the given circuit (medium).

The total electromotive force in any circuit s arising from the presence of a second circuit s' of variable current-strength J' is now given by the empirical law

$$\int p\,ds = -A^2 \frac{d}{dt}\left(J' \iint \frac{\cos(ds, ds')\,ds\,ds'}{\rho}\right),$$

where A is a constant; if the relative position of the two circuits remains unaltered, it can evidently be written

$$\int p\,ds = -A^2\frac{dJ'}{dt}\iint\frac{\cos(ds,ds')\,ds\,ds'}{\rho}.$$

This double integral is the same one that appeared in § 30; it corresponds to Neumann's electrodynamic potential (cf. formula (10, XIII.)), which, as we have seen, was only a special form of von Helmholtz's. The more general form of the given law for induced action would therefore be

$$\int p\,ds = -A^2\frac{dJ'}{dt}$$
$$\times\iint\frac{\frac{1+k}{2}\cos(ds,ds')+\frac{1-k}{2}\cos(ds,\rho)\cos(ds',\rho)}{\rho}ds\,ds'.\;\ldots(2)$$

This expression for $\int p\,ds$ forms the fundament of von Helmholtz's equations.

We next make the assumption that the above empirical laws hold not only for closed circuits but for their single elements; we can then write formula (1) as follows:

$$p=\kappa i \quad\text{or}\quad p_1+p_2=\kappa i,$$

where p_1 denotes the electrostatic force and p_2 that due to induction.

Since $p_1=-\frac{d\phi}{ds}$ and, by formula (2) and the above assumption,

$$p_2=-A^2\frac{dJ'}{dt}\int\frac{\frac{1+k}{2}\cos(ds,ds')+\frac{1-k}{2}\cos(ds,\rho)\cos(ds',\rho)}{\rho}ds',$$

it follows that

$$\kappa i=-\frac{d\phi}{ds}-A^2\frac{dJ'}{dt}$$
$$\times\int\frac{\frac{1+k}{2}\cos(ds,ds')+\frac{1-k}{2}\cos(ds,\rho)\cos(ds',\rho)}{\rho}ds'.\;\ldots(3)$$

If ρ denotes the distance between any volume-element $d\xi d\eta d\zeta$ of the region traversed by the closed circuits and the point, at which the value of the integral

$$\int \frac{J'ds'}{\rho}\left[\frac{1+k}{2}\cos(ds, ds') + \frac{1-k}{2}\cos(ds, \rho)\cos(ds', \rho)\right] \ldots (4)$$

is sought, l, m, n its direction-cosines and J_1', J_2', J_3' the components of J' along the ξ, η, ζ axes respectively, the x-component (at the point (x, y, z)) of the expression under the integral-sign arising from the component current-strength J_1' will evidently be

$$\frac{J_1'd\xi}{\rho}\left[\frac{1+k}{2}\cos(dx, d\xi) + \frac{1-k}{2}(\cos \rho, dx)\cos(\rho, d\xi)\right]$$
$$= \frac{J_1'd\xi}{\rho}\left[\frac{1+k}{2} + \frac{1-k}{2}\frac{(x-\xi)^2}{\rho^2}\right]$$

or, if we denote the component current-densities at the point (ξ, η, ζ) by u, v, w respectively,

$$\frac{u d\xi d\eta d\zeta}{\rho}\left[\frac{1+k}{2} + \frac{1-k}{2}\frac{(x-\xi)^2}{\rho^2}\right];$$

similarly, we find the following expressions for the components of the given expression due to the component current-strengths J_2' and J_3' respectively:

$$\frac{v d\xi d\eta d\zeta}{\rho}\left[\frac{1-k}{2}\frac{(x-\xi)(y-\eta)}{\rho^2}\right]$$

and

$$\frac{w d\xi d\eta d\zeta}{\rho}\left[\frac{1-k}{2}\frac{(x-\xi)(z-\zeta)}{\rho^2}\right].$$

If we denote the x-component of the given expression at the point (x, y, z) arising from all three current-strengths by dU, we have then

$$dU = \frac{d\tau}{\rho}\left\{\frac{1+k}{2}u + \frac{1-k}{2}\frac{x-\xi}{\rho^2}\Big[u(x-\xi) + v(y-\eta) + w(z-\zeta)\Big]\right\},$$

hence

$$U=\int\frac{d\tau}{\rho}\left\{\frac{1+k}{2}u+\frac{1-k}{2}\frac{x-\xi}{\rho^2}[u(x-\xi)+v(y-\eta)+w(z-\zeta)]\right\},$$

where the integration is to be extended to all volume-elements of the region traversed by the electric currents.

By the relations

$$\frac{d^2\rho}{dx\,d\xi}=-\frac{1}{\rho}+\frac{(x-\xi)^2}{\rho^3},\quad \frac{d^2\rho}{dx\,d\eta}=\frac{(x-\xi)(y-\eta)}{\rho^3}$$

and
$$\frac{d^2\rho}{dx\,d\zeta}=\frac{(x-\xi)(z-\zeta)}{\rho^3},$$

the above expression for U can be written

$$U=\bar{u}+\frac{1-k}{2}\frac{d\psi}{dx};$$

similarly we find ......(5)

$$V=\bar{v}+\frac{1-k}{2}\frac{d\psi}{dy}\quad\text{and}\quad W=\bar{w}+\frac{1-k}{2}\frac{d\psi}{dz}$$

where

$$\bar{u}=\int\frac{u\,d\tau}{\rho},\quad \bar{v}=\int\frac{v\,d\tau}{\rho},\quad \bar{w}=\int\frac{w\,d\tau}{\rho}$$

and ......(5a)
$$\psi=\int d\tau\left(u\frac{d\rho}{d\xi}+v\frac{d\rho}{d\eta}+w\frac{d\rho}{d\zeta}\right)$$

V and W are the y- and z-components respectively of integral (4) at the point (x, y, z) arising from the three component current-strengths J_1', J_2', J_3'.

We already observe the similarity between these and the preceding formulae and those of the last chapter.

If we replace the vector-equation (3) by its three component-equations and the vector-integrals of the

latter by the above values (5), we find

$$\left.\begin{aligned}\kappa u &= -\frac{d\phi}{dx} - A^2\frac{d}{dt}\left(\bar{u} + \frac{1-k}{2}\frac{d\psi}{dx}\right) = -\frac{d\phi}{dx} - A^2\frac{dU}{dt}\\ \kappa v &= -\frac{d\phi}{dy} - A^2\frac{d}{dt}\left(\bar{v} + \frac{1-k}{2}\frac{d\psi}{dy}\right) = -\frac{d\phi}{dy} - A^2\frac{dV}{dt}\\ \kappa w &= -\frac{d\phi}{dz} - A^2\frac{d}{dt}\left(\bar{w} + \frac{1-k}{2}\frac{d\psi}{dz}\right) = -\frac{d\phi}{dz} - A^2\frac{dW}{dt}\end{aligned}\right\}\ldots(6)^*$$

We next define the free electricity and the potential ϕ as in the preceding article, and formulae (19)-(23) and (26) of that article follow directly. These formulae, together with the equations of action (6), suffice to determine the behaviour of electricity in certain bodies or media; von Helmholtz not only solved the problems of the radial flow of electricity in a sphere† and its flow through an infinitely long cylinder‡ by these equations, but he employed them to examine its behaviour in the human body. There are, however, many phenomena of common occurrence that are neither contained in nor expressed by this system of equations; both the magnetic phenomena and those that appear in surrounding dielectrics, as the Hertzian oscillations, are entirely wanting here. The next step in our development is thus to introduce the necessary changes in our above equations, that is, to modify them in such a manner that they shall include all phenomena. Let us first consider their extension to dielectrics or insulators; this requires the introduction of the conception of the electric polarization (induction) of § 14, to which we refer the student here; we assume, namely, that the components of the

* Cf. von Helmholtz's works (*Wissenschaftliche Abhandlungen*), v. 1, p. 573, equations (3*b*).

† "*Radiale Strömungen der Electricität in einer leitenden Kugel,*" von Helmholtz's works, v. 1, pp. 585-599.

‡ "*Bewegung in einem unendlichen Cylinder,*" same, v. 1, pp. 603-611.

electric moment $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ are proportional to the forces that act on the electric fluids. As regards the external electromotive forces, von Helmholtz assumes, however, that they are equally divided between the galvanic and the electric polarization-currents (cf. § 14);* here the given moments will therefore be proportional to the quantities $(P+\mathfrak{X})$, $(Q+\mathfrak{Y})$ and $(R+\mathfrak{Z})$, and we must thus write

$$\mathfrak{x}'=\epsilon(P+\mathfrak{X}), \quad \mathfrak{y}'=\epsilon(Q+\mathfrak{Y}), \quad \mathfrak{z}'=\epsilon(R+\mathfrak{Z}) \quad \ldots\ldots(7)$$

For insulators the given equations must then evidently be written

$$\left.\begin{aligned} \frac{\mathfrak{x}'}{\epsilon} &= -\frac{d\phi}{dx} - A^2\frac{dU}{dt} + M_1 + \mathfrak{X} = \kappa u \\ \frac{\mathfrak{y}'}{\epsilon} &= -\frac{d\phi}{dy} - A^2\frac{dV}{dt} + M_2 + \mathfrak{Y} = \kappa v \\ \frac{\mathfrak{z}'}{\epsilon} &= -\frac{d\phi}{dz} - A^2\frac{dW}{dt} + M_3 + \mathfrak{Z} = \kappa w \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8)$$

These equations are only a more general form of equations (6). The first terms on the right represent the components of the total electromotive forces, the second those of the electric induction arising from any variation in current-strength of the electric circuits of the field, and the third, the M's, the component-forces exercised on the electric fluids of the given volume-element by the magnets (solenoids) of variable intensity; for non-magnetizable media these M's will of course vanish (cf. also pp. 370-373); $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ denote the external electromotive forces that reside within the given volume-element; these can likewise vanish. Lastly, in retaining the quantities κu, κv, κw in these equations, we are making Maxwell's assumption that the given medium is susceptible to both electric induction and galvanic conduction.

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* Cf. von Helmholtz's works, v. 1, p. 616.

If $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ are functions of the time, any variation in the electric polarization will be equivalent to an electric current, the so-called electric polarization-current of § 35. If we now assume with Maxwell that this current acts inductively like any electric current of variable current-strength, we must then include its inductive action in the terms of electric induction of the above equations (8). Since the component current-densities of this electric polarization-current are $\frac{d\mathfrak{x}'}{dt}, \frac{d\mathfrak{y}'}{dt}, \frac{d\mathfrak{z}'}{dt}$, the desired modification can be effected by replacing u, v, w in the expressions (5) for U, V, W in equations (8) by the following values respectively:

$$\left.\begin{aligned} u' &= u + \frac{d\mathfrak{x}'}{dt} = \frac{\mathfrak{x}'}{\epsilon\kappa} + \frac{d\mathfrak{x}'}{dt}, \quad v' = v + \frac{d\mathfrak{y}'}{dt} = \frac{\mathfrak{y}'}{\epsilon\kappa} + \frac{d\mathfrak{y}'}{dt} \\ \text{and} \qquad w' &= w + \frac{d\mathfrak{z}'}{dt} = \frac{\mathfrak{z}'}{\epsilon\kappa} + \frac{d\mathfrak{z}'}{dt}; \end{aligned}\right\} \ldots(9)^*$$

we have then the new relations

$$\left.\begin{aligned} \overline{u}' &= \overline{u} + \overline{\frac{d\mathfrak{x}'}{dt}} = \frac{\overline{\mathfrak{x}'}}{\epsilon\kappa} + \overline{\frac{d\mathfrak{x}'}{dt}} = \frac{1}{\kappa}(\overline{P + \mathfrak{X}}) + \epsilon\overline{\frac{d}{dt}(P + \mathfrak{X})} \\ \overline{v}' &= \overline{v} + \overline{\frac{d\mathfrak{y}'}{dt}} = \frac{\overline{\mathfrak{y}'}}{\epsilon\kappa} + \overline{\frac{d\mathfrak{y}'}{dt}} = \frac{1}{\kappa}(\overline{Q + \mathfrak{Y}}) + \epsilon\overline{\frac{d}{dt}(Q + \mathfrak{Y})} \\ \overline{w}' &= \overline{w} + \overline{\frac{d\mathfrak{z}'}{dt}} = \frac{\overline{\mathfrak{z}'}}{\epsilon\kappa} + \overline{\frac{d\mathfrak{z}'}{dt}} = \frac{1}{\kappa}(\overline{R + \mathfrak{Z}}) + \epsilon\overline{\frac{d}{dt}(R + \mathfrak{Z})}. \end{aligned}\right\} \ldots(10)$$

Equations (8) thus assume the form

$$\left.\begin{aligned} \frac{\mathfrak{x}'}{\epsilon} &= -\frac{d\phi}{dx} - A^2\frac{dU'}{dt} + M_1 + \mathfrak{X} = \kappa u = P + \mathfrak{X} \\ \frac{\mathfrak{y}'}{\epsilon} &= -\frac{d\phi}{dy} - A^2\frac{dV'}{dt} + M_2 + \mathfrak{Y} = \kappa v = Q + \mathfrak{Y} \\ \frac{\mathfrak{z}'}{\epsilon} &= -\frac{d\phi}{dz} - A^2\frac{dW'}{dt} + M_3 + \mathfrak{Z} = \kappa w = R + \mathfrak{Z}, \end{aligned}\right\} \ldots(11)$$

* Cf. von Helmholtz's works, v. 1, p. 616, equations (18a).

$$\left.\begin{array}{ll}\text{where} & U'=\overline{u}'+\dfrac{1-k}{2}\dfrac{d\psi}{dx},\quad V'=\overline{v}'+\dfrac{1-k}{2}\dfrac{d\psi}{dy}\\ \text{and} & W'=\overline{w}'+\dfrac{1-k}{2}\dfrac{d\psi}{dz}.\end{array}\right\}\ldots(12)$$

The second important change to be made in our equations is the introduction of quantities that shall explain the magnetic phenomena; for this purpose we have recourse to the conception of the magnetic polarization (cf. p. 224) and assume that the component magnetic moments λ, μ, ν (α, β, γ) per unit-volume are proportional to the forces that act on the magnetic fluids. If $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ denote the magnetic forces arising from the electric currents of the field, χ the potential of its free magnetism and θ the factor of proportionality, we have then

$$\lambda=\theta\left(\mathfrak{L}-\frac{d\chi}{dx}\right),\quad \mu=\theta\left(\mathfrak{M}-\frac{d\chi}{dy}\right),\quad \nu=\theta\left(\mathfrak{N}-\frac{d\chi}{dz}\right).\ldots(13)$$

These magnetic forces $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ can now be determined from Biot-Savart's law, which can be written in the familiar form

$$K=\frac{Ai'ds'\sin(ds,\rho)}{\rho^2}$$

(cf. formula (37, XI.)); K is the force exercised by the linear-element ds' of current-strength i' on the magnetic fluids of unit-volume-element at the distance ρ from the former. The force K_1 arising from the x-component of the given currents included in any volume-element $d\xi\, d\eta\, d\zeta=d\tau'$ of the given field will thus be

$$K_1=\frac{Au'd\tau'\sin(x,\rho)}{\rho^2}.$$

As this force acts in the yz-coordinate-plane (cf. p. 242), its components K_{11}, K_{12}, K_{13} along the x-, y-, z-axes

respectively will be

$$K_{11}=0, \quad K_{12}=\frac{Au'd\tau'(z-\zeta)}{\rho^3}=-Au'd\tau'\frac{d}{dz}\left(\frac{1}{\rho}\right),$$

$$K_{13}=\frac{-Au'd\tau'(y-\eta)}{\rho^3}=Au'd\tau'\frac{d}{dy}\left(\frac{1}{\rho}\right);$$

similarly, analogous expressions can be found for the components of the forces K_2 and K_3 arising from the y- and z-components respectively of the given electric currents. We tabulate the values found for these nine component-forces as follows:

Per volume-element $d\tau'$.			
Arising from current-components.	$\mathfrak{L}$	$\mathfrak{M}$	$\mathfrak{N}$
u'	$K_{11}=0$	$K_{12}=-Au'd\tau'\frac{d}{dz}\left(\frac{1}{\rho}\right)$	$K_{13}=Au'd\tau'\frac{d}{dy}\left(\frac{1}{\rho}\right)$
v'	$K_{21}=Av'd\tau'\frac{d}{dz}\left(\frac{1}{\rho}\right)$	$K_{22}=0$	$K_{23}=-Av'd\tau'\frac{d}{dx}\left(\frac{1}{\rho}\right)$
w'	$K_{31}=-Aw'd\tau'\frac{d}{dy}\left(\frac{1}{\rho}\right)$	$K_{32}=Aw'd\tau'\frac{d}{dx}\left(\frac{1}{\rho}\right)$	$K_{33}=0$

(14)

Compare the investigations on p. 243 for the determination of the signs.

This table evidently gives the following values for $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, the magnetic forces due to the action of all the currents of the field:

$$\left.\begin{aligned}\mathfrak{L}&=A\int\left[v'\frac{d}{dz}\left(\frac{1}{\rho}\right)-w'\frac{d}{dy}\left(\frac{1}{\rho}\right)\right]d\tau'\\ \mathfrak{M}&=A\int\left[w'\frac{d}{dx}\left(\frac{1}{\rho}\right)-u'\frac{d}{dz}\left(\frac{1}{\rho}\right)\right]d\tau'\\ \mathfrak{N}&=A\int\left[u'\frac{d}{dy}\left(\frac{1}{\rho}\right)-v'\frac{d}{dx}\left(\frac{1}{\rho}\right)\right]d\tau'\end{aligned}\right\}\quad\ldots\ldots\ldots(15)$$

$$\left.\begin{array}{ll}\text{or} & \mathfrak{L}=A\left(\frac{d\bar{v}'}{dz}-\frac{d\bar{w}'}{dy}\right),\quad \mathfrak{M}=A\left(\frac{d\bar{w}'}{dx}-\frac{d\bar{u}'}{dz}\right)\\ \text{and} & \mathfrak{N}=A\left(\frac{d\bar{u}'}{dy}-\frac{d\bar{v}'}{dx}\right),\\ \text{where} & \bar{u}'=\int\frac{u'd\tau'}{\rho},\quad \bar{v}'=\int\frac{v'd\tau'}{\rho},\quad \bar{w}'=\int\frac{w'd\tau'}{\rho}.\end{array}\right\}\ldots(15a)$$

Substitute these values in formulae (13) and we get

$$\left.\begin{array}{l}\frac{\lambda}{\theta}=-\frac{d\chi}{dx}+A\left(\frac{d\bar{v}'}{dz}-\frac{d\bar{w}'}{dy}\right),\quad \frac{\mu}{\theta}=-\frac{d\chi}{dy}+A\left(\frac{d\bar{w}'}{dx}-\frac{d\bar{u}'}{dz}\right)\\ \text{and}\qquad \frac{\nu}{\theta}=-\frac{d\chi}{dz}+A\left(\frac{d\bar{u}'}{dy}-\frac{d\bar{v}'}{dx}\right)\end{array}\right\}(16)$$

or

$$\left.\begin{array}{l}\frac{\lambda}{\theta}=-\frac{d\chi}{dx}+A\left(\frac{dV'}{dz}-\frac{dW'}{dy}\right)\\ \frac{\mu}{\theta}=-\frac{d\chi}{dy}+A\left(\frac{dW'}{dx}-\frac{dU'}{dz}\right)\\ \frac{\nu}{\theta}=-\frac{d\chi}{dz}+A\left(\frac{dU'}{dy}-\frac{dV'}{dx}\right)\end{array}\right\}\ldots\ldots\ldots\ldots(16a)$$

Lastly, we come to the determination of the quantities M of formulae (11), the forces exercised by the magnets of variable intensity on the electric currents. This induced action between the magnets and the currents of the field must now follow from the expression for the electrodynamic potential. In the present case it is immaterial whether we use von Helmholtz's expression for this potential, which would be the more consistent with the following development, or the simpler one employed by Neumann; this becomes evident upon examining the ensuing derivation.

The induced action between any current i' and unit linear element of unit-current-strength is given by the expression

$$-\frac{d}{dt}Ai'\int\frac{\cos(ds,ds')ds'}{\rho}\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

(cf. formula (10, XIII.))

$\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ were the forces that gave rise to the displacement of the magnetic fluids in unit-volume-element; the work done by these forces per volume-element $dx\,dy\,dz$ must then have been

$$\mathfrak{L}\lambda\,dy\,dz\,.\,dx=\lambda\mathfrak{L}\,d\tau,\quad \mu\mathfrak{M}\,d\tau \text{ and } \nu\mathfrak{N}\,d\tau,$$

and hence the total work done in bringing about the given magnetic state

$$\int(\lambda\mathfrak{L}+\mu\mathfrak{M}+\nu\mathfrak{N})\,d\tau. \quad\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

This quantity is now only another expression for the electrodynamic potential. We replace here $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ by their values from formulae (15), and we have

$$\begin{aligned}\int(\lambda\mathfrak{L}+\mu\mathfrak{M}+\nu\mathfrak{N})\,d\tau=A\int\Big\{\lambda\int&\Big[v'\frac{d}{dz}\Big(\frac{1}{\rho}\Big)-w'\frac{d}{dy}\Big(\frac{1}{\rho}\Big)\Big]d\tau'\\ +\mu\int&\Big[w'\frac{d}{dx}\Big(\frac{1}{\rho}\Big)-u'\frac{d}{dz}\Big(\frac{1}{\rho}\Big)\Big]d\tau'\\ +\nu\int&\Big[u'\frac{d}{dy}\Big(\frac{1}{\rho}\Big)-v'\frac{d}{dx}\Big(\frac{1}{\rho}\Big)\Big]d\tau'\Big\}d\tau.\end{aligned}$$

This integral can be separated into the three following component-integrals, arising from the component current-strengths u', v', w':

$$A\int\Big[\nu\int u'\frac{d}{dy}\Big(\frac{1}{\rho}\Big)d\tau'-\mu\int u'\frac{d}{dz}\Big(\frac{1}{\rho}\Big)d\tau'\Big]d\tau,$$

$$A\int\Big[\lambda\int v'\frac{d}{dz}\Big(\frac{1}{\rho}\Big)d\tau'-\nu\int v'\frac{d}{dx}\Big(\frac{1}{\rho}\Big)d\tau'\Big]d\tau,$$

$$A\int\Big[\mu\int w'\frac{d}{dx}\Big(\frac{1}{\rho}\Big)d\tau'-\lambda\int w'\frac{d}{dy}\Big(\frac{1}{\rho}\Big)d\tau'\Big]d\tau,$$

which, upon reversing the order of integration and employing the relations

$$\frac{d}{dx}\Big(\frac{1}{\rho}\Big)=-\frac{d}{d\xi}\Big(\frac{1}{\rho}\Big),\quad \frac{d}{dy}\Big(\frac{1}{\rho}\Big)=-\frac{d}{d\eta}\Big(\frac{1}{\rho}\Big),\quad \frac{d}{dz}\Big(\frac{1}{\rho}\Big)=-\frac{d}{d\zeta}\Big(\frac{1}{\rho}\Big),$$

can be written as follows:

$$A\int u'd\tau'\left\{\frac{d}{d\zeta}\int\frac{\mu d\tau}{\rho}-\frac{d}{d\eta}\int\frac{\nu d\tau}{\rho}\right\}=A\int u'd\tau'\left(\frac{dM}{d\zeta}-\frac{dN}{d\eta}\right)$$

$$A\int v'd\tau'\left\{\frac{d}{d\xi}\int\frac{\nu d\tau}{\rho}-\frac{d}{d\zeta}\int\frac{\lambda d\tau}{\rho}\right\}=A\int v'd\tau'\left(\frac{dN}{d\xi}-\frac{dL}{d\zeta}\right)$$

$$A\int w'd\tau'\left\{\frac{d}{d\eta}\int\frac{\lambda d\tau}{\rho}-\frac{d}{d\xi}\int\frac{\mu d\tau}{\rho}\right\}=A\int w'd\tau'\left(\frac{dL}{d\eta}-\frac{dM}{d\xi}\right).$$

These expressions reduce to the following for unit-linear-elements of current-strength i' at any point (x, y, z):

$$i'A\left(\frac{dM}{dz}-\frac{dN}{dy}\right),\quad i'A\left(\frac{dN}{dx}-\frac{dL}{dz}\right),\quad i'A\left(\frac{dL}{dy}-\frac{dM}{dx}\right);$$

the latter evidently correspond to the values assumed by the integral

$$i'A\int\frac{\cos(ds, ds')ds'}{\rho}$$

of expression (17) for the components of the given current along the x, y, z-axes respectively. It thus follows that the induced action of magnets (solenoids) on the given linear elements, $dx=dy=dz=1$, will be

$$-\frac{d}{dt}i'A\left(\frac{dM}{dz}-\frac{dN}{dy}\right),\quad -\frac{d}{dt}i'A\left(\frac{dN}{dx}-\frac{dL}{dz}\right),$$
$$-\frac{d}{dt}i'A\left(\frac{dL}{dy}-\frac{dM}{dx}\right),$$

which for $i'=1$ evidently give the desired expressions for M_1, M_2, M_3 respectively.

We have therefore

$$\left.\begin{aligned} M_1=A\frac{d}{dt}\left(\frac{dN}{dy}-\frac{dM}{dz}\right),\quad M_2=A\frac{d}{dt}\left(\frac{dL}{dz}-\frac{dN}{dx}\right),\\ M_3=A\frac{d}{dt}\left(\frac{dM}{dx}-\frac{dL}{dy}\right),\end{aligned}\right\}\quad(19a)^*$$

* Cf. von Helmholtz's works, v. 1, p. 619, equations (19*a*).

where
$$\left.\begin{aligned} L=\iiint\frac{\lambda}{\rho}d\xi d\eta d\zeta,\quad M=\iiint\frac{\mu}{\rho}d\xi d\eta d\zeta,\\ N=\iiint\frac{\nu}{\rho}d\xi d\eta d\zeta.\end{aligned}\right\}\text{......(20)*}$$

Finally, substitute these values for M_1, M_2, M_3 in equations (11), and we find

$$\left.\begin{aligned} \frac{\mathfrak{x}'}{\epsilon}=\kappa u=-\frac{d\phi}{dx}-A^2\frac{dU'}{dt}+A\frac{d}{dt}\left(\frac{dN}{dy}-\frac{dM}{dz}\right)+\mathfrak{X}\\ \frac{\mathfrak{y}'}{\epsilon}=\kappa v=-\frac{d\phi}{dy}-A^2\frac{dV'}{dt}+A\frac{d}{dt}\left(\frac{dL}{dz}-\frac{dN}{dx}\right)+\mathfrak{Y}\\ \frac{\mathfrak{z}'}{\epsilon}=\kappa w=-\frac{d\phi}{dz}-A^2\frac{dW'}{dt}+A\frac{d}{dt}\left(\frac{dM}{dx}-\frac{dL}{dy}\right)+\mathfrak{Z}\end{aligned}\right\}\text{(21)}$$

(cf. formulae (28, XVI.)).

The six equations of action (16) and (21) are identical to those ((28) and (29)) of the preceding article; the two remaining equations of the given system are the conditions (30, XVI.) and (31, XVI.), obtained from the definitions of the real electricity and magnetism.

The assumption of a dielectric and the introduction of this conception into von Helmholtz's original equations of electric action (6) have led to the more general system of equations (21), which differ materially in one respect from the former; the changes in question can best be brought to light by comparing the special forms assumed by each system for the state of the ether in a conductor, whose conductivity is so large, that is, whose κ is so small, that the action of its electric polarization may be neglected in comparison to that of the electric currents traversing it, so that the terms $\frac{d\mathfrak{x}'}{dt}, \frac{d\mathfrak{y}'}{dt}, \frac{d\mathfrak{z}'}{dt}$ in the expressions for U', V', W' of equations (21) may be rejected in comparison to the terms $\frac{\mathfrak{x}'}{\epsilon\kappa}, \frac{\mathfrak{y}'}{\epsilon\kappa}, \frac{\mathfrak{z}'}{\epsilon\kappa}$ respectively

* Cf. von Helmholtz's works, v. 1, p. 619, equations (19*c*).

(cf. formulae (9) and text p. 364); $\overline{u}'$, $\overline{v}'$, $\overline{w}'$ then become identical to $\overline{u}$, $\overline{v}$, $\overline{w}$ respectively (cf. formulae (10)) and hence U', V', W' to U, V, W (cf. formulae (5) and (12)), and equations (21) thus reduce to the following simpler ones, after we have put $\mathfrak{X}=\mathfrak{Y}=\mathfrak{Z}=0$:

$$\left.\begin{aligned}\kappa u &= -\frac{d\phi}{dx}+A^2\frac{dU}{dt}+A\frac{d}{dt}\left(\frac{dN}{dy}-\frac{dM}{dz}\right)\\ \kappa v &= -\frac{d\phi}{dy}+A^2\frac{dV}{dt}+A\frac{d}{dt}\left(\frac{dL}{dz}-\frac{dN}{dx}\right)\\ \kappa w &= -\frac{d\phi}{dz}+A^2\frac{dW}{dt}+A\frac{d}{dt}\left(\frac{dM}{dx}-\frac{dL}{dy}\right)\end{aligned}\right\}\ \ldots\ldots(22)$$

The only difference between these equations and equations (6) is that the former contain terms that express an induced action between the magnetic field and its electric fluids (cf. text p. 367), which corresponds to the assumption that the given medium is susceptible to magnetic polarization, whereas such terms are entirely wanting in the latter. To compare these two systems of equations several transformations will be found necessary: we perform the ∇^2-operation on the first of equations (6), and we get

$$\begin{aligned}\kappa\nabla^2 u &= -\frac{d\nabla^2\phi}{dx}-A^2\frac{d\nabla^2 U}{dt}\\ &= -\frac{d\nabla^2\phi}{dx}-A^2\frac{d}{dt}\left(\nabla^2\overline{u}+\frac{1-k}{2}\,\frac{d\nabla^2\psi}{dx}\right)\end{aligned}$$

or, since $\nabla^2\overline{u}=-4\pi u$ and $\nabla^2\psi=2\epsilon\dfrac{d\phi}{dt}$

(cf. formula (21, XVI.)),

$$\kappa\nabla^2 u=4\pi A^2\frac{du}{dt}-\frac{d}{dx}\left\{\nabla^2\phi+(1-k)\epsilon A^2\frac{d^2\phi}{dt^2}\right\}\ \ldots(23)^*$$

and similar equations for v and w.

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* Cf. von Helmholtz's works, v. 1, p. 626.

By a similar treatment the first of equations (22) can be written

$$\kappa\nabla^2 u = 4\pi A^2\frac{du}{dt} - \frac{d}{dx}\left\{\nabla^2\phi + (1-k)\epsilon A^2\frac{d^2\phi}{dt^2}\right\}$$

$$+A\frac{d}{dt}\left(\frac{d\nabla^2 N}{dy} - \frac{d\nabla^2 M}{dz}\right). \quad \text{.........(24)}$$

To evaluate the last bracket we replace $\nabla^2 M$ and $\nabla^2 N$ by their values $-4\pi\mu$ and $-4\pi\nu$ respectively, then μ and ν by their values from formulae (16), which assume here the simpler form

$$\frac{\mu}{\theta} = -\frac{d\chi}{dy} + A\left(\frac{d\bar{w}}{dx} - \frac{d\bar{u}}{dz}\right) \text{ and } \frac{\nu}{\theta} = -\frac{d\chi}{dz} + A\left(\frac{d\bar{u}}{dy} - \frac{d\bar{v}}{dx}\right),$$

and we get

$$\frac{d\nabla^2 N}{dy} - \frac{d\nabla^2 M}{dz} = 4\pi\left(\frac{d\mu}{dz} - \frac{d\nu}{dy}\right)$$

$$= -4\pi\theta A\left[\nabla^2\bar{u} - \frac{d}{dx}\left(\frac{d\bar{u}}{dx} + \frac{d\bar{v}}{dy} + \frac{d\bar{w}}{dz}\right)\right]$$

or, lastly, by the relations

$$\nabla^2\bar{u} = -4\pi u \text{ and } \frac{d\bar{u}}{dx} + \frac{d\bar{v}}{dy} + \frac{d\bar{w}}{dz} = -\epsilon\frac{d\phi}{dt}$$

(cf. formula (23, XVI.)),

$$\frac{d\nabla^2 N}{dy} - \frac{d\nabla^2 M}{dz} = 4\pi\theta A\left(4\pi u - \epsilon\frac{d^2\phi}{dx\,dt}\right).$$

Equation (24) can thus be written

$$\kappa\nabla^2 u = (1+4\pi\theta)4\pi A^2\frac{du}{dt}$$

$$-\frac{d}{dx}\left\{\nabla^2\phi + (1+4\pi\theta-k)\epsilon A^2\frac{d^2\phi}{dt^2}\right\}. \quad \text{.....(25)*}$$

A comparison of the two equations (23) and (25) for u shows that the coefficients of the former undergo slight

* Cf. von Helmholtz's works, v. 1, p. 626.

alterations upon assuming a dielectric: A^2 is namely transformed into $(1+4\pi\theta)A^2$ and hence k into $\frac{k}{1+4\pi\theta}$; observe, moreover, that the constant ε plays here no rôle whatever. It thus follows that, when the given medium is assumed to be susceptible to magnetic induction, the value of the constant k apparently decreases in the ratio $1:(1+4\pi\theta)$, whereas that of the constant A^2 is apparently multiplied by $(1+4\pi\theta)$. The value apparently found for A^2 from experiments taken in any medium, whose constant of magnetic induction is θ, will be therefore not its real value but that of $(1+4\pi\theta)A^2$.

It is possible to determine experimentally the ratio between the values of the constant $(1+4\pi\theta)$ for two different bodies or media or the value of this constant referred to the ether of empty space or a vacuum, but not its absolute value. It is thus quite immaterial whether we put $1+4\pi\theta=1$ in a vacuum, as Poisson and others have done, or not. The foregoing is, of course, also true of the constant $(\varepsilon+4\pi\epsilon)$, where ϵ denotes the constant of electric induction (cf. also § 17). On the other hand there are several scientists who put $(1+4\pi\epsilon)>1\,(\varepsilon=1)$ in a vacuum, in order to avoid negative values of ϵ in bodies susceptible to magnetic induction (cf. p. 131).

SECTION XXXIX. TREATMENT OF FAMILIAR PROBLEMS BY VON HELMHOLTZ'S EQUATIONS OF ACTION AT A DISTANCE AND EXAMINATION OF THE MEDIUM-CONSTANTS. SIMPLEST FORM OF VON HELMHOLTZ'S EQUATIONS; THEIR SPECIAL FORM ON THE DIVIDING-SURFACES OF ADJOINING MEDIA.

The phenomena peculiar to von Helmholtz's equations of action at a distance and an interpretation of his medium-constants can best be found by an examination of special cases or problems from the theory of

electrostatics. The electrostatic state of the ether is characterized by the conditions

$$u=v=w=\frac{du}{dt}=\frac{dv}{dt}=\frac{dw}{dt}=0 \text{ and } \mathfrak{X}=\mathfrak{Y}=\mathfrak{Z}=0,$$

where, as above, u, v, w denote the component-current-strengths of the given galvanic currents.

Our equations of electric action (21) thus reduce to the following for the electrostatic state:

$$\frac{\mathfrak{x}'}{\epsilon}=-\frac{d\phi}{dx},\quad \frac{\mathfrak{y}'}{\epsilon}=-\frac{d\phi}{dy},\quad \frac{\mathfrak{z}'}{\epsilon}=-\frac{d\phi}{dz}. \quad\text{..........(26)}$$

for insulators or dielectrics and to

$$0=\kappa u=-\frac{d\phi}{dx},\quad 0=\kappa v=-\frac{d\phi}{dy},\quad 0=\kappa w=-\frac{d\phi}{dz}$$

for conductors; from the latter it follows that ϕ is constant in conductors.

For homogeneous insulators equations (26) give

$$\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}=-\epsilon\nabla^2\phi. \quad\text{..............(27)}$$

The potential function ϕ at any point of space is now always given by the integral-expression

$$\phi=\frac{1}{\mathfrak{e}}\int\frac{\epsilon_f'\,d\tau}{\rho}, \text{ hence } \nabla^2\phi=-\frac{4\pi}{\mathfrak{e}}\epsilon_f', \quad\text{.........(28)}$$

where $$\epsilon_f'=\epsilon_r'+\epsilon_p';$$

for the electrostatic state of the ether ϵ_r' and ϵ_p' are evidently given by the following expressions:

$$\frac{d\epsilon_r'}{dt}=-\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0, \text{ hence } \epsilon_r'=\text{const.}=\epsilon_r^0, \quad\text{...(29)}$$

where ϵ_r^0 denotes the initial density of the real electricity, and

$$\epsilon_p'=-\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right),$$

which by formula (27) can be written

$$\epsilon_p' = \epsilon \nabla^2 \phi. \quad \text{......................(30)}$$

We thus have for the electrostatic state

$$\epsilon_f' = \epsilon_r^0 + \epsilon_p',$$

which for $\epsilon_r^0 = 0$ reduces to

$$\epsilon_f' = \epsilon_p';$$

upon replacing here ϵ_f' and ϵ_p' by their values from formulae (28) and (30) respectively, we get

$$-\frac{\mathfrak{k}}{4\pi}\nabla^2\phi = \epsilon\nabla^2\phi$$

or
$$\left(\epsilon + \frac{\mathfrak{k}}{4\pi}\right)\nabla^2\phi = 0;$$

since ϵ and $\mathfrak{k}$ are both positive quantities, it thus follows that

$$\nabla^2\phi = 0,$$

hence
$$\epsilon_f' = \epsilon_p' = 0; \quad \text{......................(31)}$$

that is, electricity is neither created nor destroyed by electric induction at any point, where real electricity has not initially resided.

Similarly, we find that for $\epsilon_r^0 \gtrless 0$

$$\nabla^2\phi = -\frac{\epsilon_r^0}{\epsilon + \frac{\mathfrak{k}}{4\pi}}$$

or, upon replacing $\nabla^2\phi$ by its value (28),

$$\epsilon_f' = \frac{\epsilon_r^0}{1 + \frac{4\pi\epsilon}{\mathfrak{k}}}; \quad \text{.....................(32)}$$

that is, electricity is rendered inactive by electric induction at every point, where real electricity has initially resided.

It follows from this value for ϵ_f' that

$$\epsilon_p' = \epsilon_f' - \epsilon_r^0 = \frac{-\dfrac{4\pi\epsilon}{\mathfrak{e}}}{1+\dfrac{4\pi\epsilon}{\mathfrak{e}}}\,\epsilon_r^0;\ \ldots\ldots\ldots\ldots\ldots(33)$$

that is, ϵ_p' and ϵ_r^0 are always proportional to each other.

It follows, moreover, from formula (32) that for $\epsilon/\mathfrak{e} = \infty$

$$\epsilon_f' = \epsilon_r^0 + \epsilon_p' = 0;$$

that is, here no electricity could accumulate in the interior of the given insulator, it could, however, appear on its surface; in this case the insulator would thus behave similarly to a conductor.

Let us next examine a special case of the above general problem, namely the state of an homogeneous dielectric, in which a small metallic sphere, upon whose surface the quantity of real electricity e_r^0 resides, is inserted. If we assume that $\epsilon_r^0 = 0$ at every point of our dielectric proper, we have then by formula (31)

$$\epsilon_f' = \epsilon_p' = 0.$$

The value of the potential ϕ at any point of the dielectric, whose distance ρ from the centre of the given sphere is large in comparison to its radius, can be written

$$\phi = \frac{1}{\mathfrak{e}\rho}\int \epsilon_f' d\tau = \frac{1}{\mathfrak{e}}\,\frac{e_f'}{\rho} = \frac{1}{\mathfrak{e}}\,\frac{e_r^0 + e_p'}{\rho},$$

a familiar expression.

We next replace ϕ by this value in formulae (26), and we have

$$\frac{\mathfrak{x}'}{\epsilon} = \frac{e_p' x}{\mathfrak{e}\rho^3},\quad \frac{\mathfrak{y}'}{\epsilon} = \frac{e_p' y}{\mathfrak{e}\rho^3},\quad \frac{\mathfrak{z}'}{\epsilon} = \frac{e_p' z}{\mathfrak{e}\rho^3},$$

which give the following value for the resultant electric polarization $\mathfrak{R}$ at the given point:

$$\mathfrak{R} = \sqrt{\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2} = \frac{\epsilon e_f'}{\mathfrak{e}\rho^2}.\ \ldots\ldots\ldots\ldots\ldots\ldots(34)$$

To interpret this expression, we describe two concentric spheres of radii ρ and $\rho + d\rho$ about the centre of the metallic sphere: as $\mathfrak{M}$ acts radially on the volume-elements $d\rho do$ of this spherical shell, the electric moment of every one of these elements will thus be

$$\mathfrak{M} d\rho do.$$

We can now conceive this moment as due to the action of the quantities of electricity—$\mathfrak{M} do$ on the inner and $\mathfrak{M} do$ on the outer surface-element do of the given shell, and hence the total electric moment of the given shell as due to the accumulation of the quantities of electricity $-4\pi\rho^2\mathfrak{M} = -\frac{4\pi\epsilon e_p'}{\mathfrak{e}}$ on its inner surface and $4\pi\rho^2\mathfrak{M} = \frac{4\pi\epsilon e_p}{\mathfrak{e}}$ on its outer; we observe that these final expressions are entirely independent of the vector ρ. If we thus conceive the whole surrounding dielectric to be divided up into such spherical shells, upon everyone of whose inner surfaces the given (constant) quantity of negative electricity and upon everyone of whose outer ones the same given (constant) quantity of positive electricity resides, the action of the positive electricity on the outer surface of any given shell will evidently cancel that of the negative electricity on the inner surface of its neighbouring one; it is, moreover, evident that the action of the positive electricity on the outer surface of the spherical shell at infinite distance from the metallic sphere can be neglected. The only quantity of electricity whose action concerns us here is thus that on the inner surface of the innermost spherical shell, the one whose inner surface is that of the metallic sphere itself. We have therefore

$$e_p' = -\frac{4\pi\epsilon}{\mathfrak{e}} e_f',$$

hence $$e_f' = e_r^0 + e_p' = e_r^0 - \frac{4\pi\epsilon}{\mathfrak{e}} e_f',$$

which gives

$$e_f'=\frac{e_r^0}{1+\dfrac{4\pi\epsilon}{\mathfrak{e}}}\quad\text{and}\quad e_p'=\frac{-\dfrac{4\pi\epsilon}{\mathfrak{e}}}{1+\dfrac{4\pi\epsilon}{\mathfrak{e}}}e_r^0,\text{..........}(35)$$

the formulae just found for the general problem. The treatment of this special case was only to offer the student an illustrative interpretation for the given expressions.

Another special case of the general problem, which is of particular interest, is that where two small metallic spheres are charged with electricity and inserted in a dielectric. We denote the initial electric charge on the first sphere by e_{1r}^0 and that on the second by e_{2r}^0, and assume that before their insertion $e_r^0=0$ at every point of the dielectric. We know then from the preceding problem that the charge on each sphere gives rise to a radial electric polarization in the surrounding medium, whereby, however, free electricity is neither created nor destroyed, except on the surfaces of the spheres themselves; here we have by formulae (35)

$$e_{1p}=\frac{-\dfrac{4\pi\epsilon}{\mathfrak{e}}}{1+\dfrac{4\pi\epsilon}{\mathfrak{e}}}e_{1r}^0\quad\text{and}\quad e_{2p}=\frac{-\dfrac{4\pi\epsilon}{\mathfrak{e}}}{1+\dfrac{4\pi\epsilon}{\mathfrak{e}}}e_{2r}^0.$$

The force exercised by the free electricity $e_{1f}=e_{1r}^0+e_{1p}$ of the first sphere on unit-quantity of electricity would thus be

$$\frac{e_{1r}^0+e_{1p}}{\mathfrak{e}\rho^2}=\frac{e_{1r}^0}{\mathfrak{e}\left(1+\dfrac{4\pi\epsilon}{\mathfrak{e}}\right)\rho^2}.$$

The force exercised by the first sphere on the second is not, however, given by the expression

$$\frac{e_{1r}^0e_{2f}}{\mathfrak{e}\left(1+\dfrac{4\pi\epsilon}{\mathfrak{e}}\right)\rho^2}=\frac{e_{1r}^0e_{2r}^0}{\mathfrak{e}\left(1+\dfrac{4\pi\epsilon}{\mathfrak{e}}\right)^2\rho^2},\text{............}(36)$$

as we should expect, but by the simpler one

$$\frac{e_{1r}{}^0 e_{2r}{}^0}{\mathfrak{e}\left(1+\frac{4\pi\epsilon}{\mathfrak{e}}\right)\rho^2}=\frac{e_{1r}{}^0 e_{2r}{}^0}{(\mathfrak{e}+4\pi\epsilon)\rho^2}; \qquad (37)$$

that is, the force exercised by the first sphere on the second is entirely independent of e_{2p}, the quantity of electricity due to electric induction on the surface of the latter. The expression for the action of the second sphere on the first is likewise obtained by excluding the action of e_{1p}, the electricity due to electric induction on the surface of the first; it is evidently given by the same expression. This law (37) is confirmed by empirical results; von Helmholtz's attempt* to explain it has already been discussed at the end of § 17. The law expressed by formula (36) corresponds to that given by formula (14, VII.) of Maxwell's theory, whereas formula (13, VII.) is Maxwell's form of formula (37).

$e_{1r}{}^0$ and $e_{2r}{}^0$ of formula (37) denote quantities of real electricity measured in the electrostatic units of the given medium, air; if we define unit-force as the force exercised by two such unit-quantities of real electricity at unit-distance apart on each other, formula (37) will assume the special form

$$1=\frac{1}{\mathfrak{e}+4\pi\epsilon} \quad \text{or} \quad \mathfrak{e}=1-4\pi\epsilon, \qquad (38)$$

whereby the arbitrary constant $\mathfrak{e}$ is determined; its value will thus depend on the constitution of the given medium or dielectric. If we assume that air is unsusceptible to electric induction and thus put $\epsilon_a=0$, we have $\mathfrak{e}=1$, values that characterize the old theory of electricity and magnetism.

If we retain the above definition of unit-force but

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* Cf. also von Helmholtz's works, v. 1, p. 614.

measure the given quantities of real electricity in magnetic units, formula (38) must evidently be written

$$1=\frac{1}{\mathfrak{V}^2(\mathfrak{e}+4\pi\epsilon)} \quad \text{or} \quad \mathfrak{e}=\frac{1-4\pi\epsilon\mathfrak{V}^2}{\mathfrak{V}^2}, \quad \ldots\ldots\ldots(39)$$

which is quite a different value for $\mathfrak{e}$ from the above (38).

Formula (37) shows that the mutual action between two quantities of real electricity upon being inserted in a dielectric is diminished in a constant ratio, whose value is alone determined by that of the medium constant ϵ. Only the ratio of the values of the quantity $(\mathfrak{e}+4\pi\epsilon)$ for two different media and not its absolute value for any given medium can, however, be determined experimentally (cf. p. 373). As the value found for the electrostatic unit in any dielectric, as air, will thus be $\sqrt{\mathfrak{e}+4\pi\epsilon}$ times too large, both the current-electricity and the current-units themselves will be $\sqrt{\mathfrak{e}+4\pi\epsilon}$ times too small. We know now that the constant A^2 is proportional to the electrodynamic force that acts between two electrostatic units of current (cf. also § 30). It thus follows that the value found for the constant A in any dielectric, as air, is not its real but only its apparent value; the former would be that assumed by it in a medium that is unsusceptible to electric induction, namely $\frac{\mathfrak{A}}{\sqrt{\mathfrak{e}+4\pi\epsilon}}$, where $\mathfrak{A}$ denotes its apparent value and ϵ the constant of electric induction of the given dielectric—we neglect here the effect of magnetic induction on the value apparently assumed by A^2.

A comparison of the present results with those on p. 373 shows that the real value of the constant A must evidently be given by the expression

$$A=\frac{1}{\mathfrak{A}\sqrt{\mathfrak{e}+4\pi\epsilon}\sqrt{1+4\pi\theta}}, \quad \ldots\ldots\ldots\ldots(40)^*$$

* Cf. von Helmholtz's works, v. 1, p. 627.

where $\mathfrak{A}$ denotes the value of $\frac{1}{A}$ determined by experiment in the given dielectric and ϵ and θ its constants of electric and magnetic induction respectively.

Lastly, let us examine the following special form of our general problem: two infinitely large metal sheets are inserted in a dielectric parallel to each other and at the finite distance d apart, the one is maintained at a constant potential $\phi = a$ and the other is connected with the earth ($\phi = 0$); such a system is known as a condenser. If we assume that $\epsilon_r{}^0 = 0$ at every point of the intervening dielectric, we have, as above,

$$\epsilon_f' = \epsilon_p' = 0, \text{ hence } \nabla^2\phi = 0$$

at every point of it. If we lay the origin of our system of coordinates in the inner surface of the sheet connected with the earth and its x-axis at right angles to it, all quantities will be approximately functions of x only. The condition $\nabla^2\phi = 0$ then reduces to

$$\frac{d^2\phi}{dx^2} = 0, \text{ hence } \phi = \frac{ax}{d}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

$\nabla^2\phi$ also vanishes at every point in the interior of the metal sheets. The inner surfaces of the two sheets are thus the only regions where real electricity can accumulate; here the following general formula will therefore evidently hold:

$$\epsilon_r' = \epsilon_f' - \epsilon_p' = -\left(\epsilon + \frac{\mathfrak{k}}{4\pi}\right)\nabla^2\phi$$

(cf. formulae (28) and (30)). The surface-density E_r' of the real electricity at any point on the surface of either sheet is thus given by the following integral (cf. § 7):

$$E_r' = \int_0^\delta \epsilon_r dx = -\left(\epsilon + \frac{\mathfrak{k}}{4\pi}\right)\int_0^\delta \nabla^2\phi\, dx.$$

Since $\nabla^2\phi$ reduces here to $\frac{d^2\phi}{dx^2}$, we can write

$$E_r' = -\left(\epsilon+\frac{\mathfrak{k}}{4\pi}\right)\int_0^\delta d\left(\frac{d\phi}{dx}\right) = -\left(\epsilon+\frac{\mathfrak{k}}{4\pi}\right)\left[\left(\frac{d\phi}{dx}\right)_\delta - \left(\frac{d\phi}{dx}\right)_0\right];$$

$\left(\frac{d\phi}{dx}\right)_0$ vanishes, since $\phi=0$ in the given region, whereas by formula (41)

$$\left(\frac{d\phi}{dx}\right)_\delta = \frac{a}{d}.$$

Hence we have

$$E_r' = -\left(\epsilon+\frac{\mathfrak{k}}{4\pi}\right)\frac{a}{d}.$$

The capacity C of the given condenser is therefore

$$C=\left(\epsilon+\frac{\mathfrak{k}}{4\pi}\right)\frac{s}{d}=\left(1+\frac{4\pi\epsilon}{\mathfrak{k}}\right)\frac{\mathfrak{k}s}{4\pi d}, \quad \text{............(42)}$$

where s denotes the area of either sheet (cf. also § 23). If the intervening dielectric is air, we have

$$C_a=\left(1+\frac{4\pi\epsilon_a}{\mathfrak{k}}\right)\frac{\mathfrak{k}s}{4\pi d}. \quad \text{................(42a)}$$

These last two formulae give

$$\frac{C}{C_a}=\frac{1+\frac{4\pi\epsilon}{\mathfrak{k}}}{1+\frac{4\pi\epsilon_a}{\mathfrak{k}}}; \quad \text{....................(43)}$$

as this quotient can evidently be determined experimentally, we obtain hereby a meaning for the quantity $\frac{\epsilon}{\mathfrak{k}}$ (cf. also p. 194).

In the preceding problem we have determined the value of E_r' on the surface of a condenser of given configuration; the general expressions for E_r', E_f', etc. on

the dividing surfaces of adjoining media of any configuration, in fact, the special form assumed by von Helmholtz's equations of electricity and magnetism (cf. p. 390) for such surfaces or films, can easily be found by assuming the principle of the continuity of transitions (cf. p. 19) and proceeding as in §§ 5, 7 and 15. As illustration let the condition for aphotic motion on the dividing-surface of two dielectrics suffice here. If we exclude the action of electric forces arising from friction from the given film, we have

$$\epsilon_r' = \epsilon_f' - \epsilon_p' = 0,$$

which by formulae (28) and (30) can be written

$$\epsilon_f' - \epsilon_p' = -\frac{\mathfrak{k}}{4\pi}\nabla^2\phi - \left[\frac{d}{dx}\left(\epsilon\frac{d\phi}{dx}\right) + \frac{d}{dy}\left(\epsilon\frac{d\phi}{dy}\right) + \frac{d}{dz}\left(\epsilon\frac{d\phi}{dz}\right)\right] = 0$$

or
$$\frac{d}{dx}\left[\left(1+\frac{4\pi\epsilon}{\mathfrak{k}}\right)\frac{d\phi}{dx}\right] + \frac{d}{dy}\left[\left(1+\frac{4\pi\epsilon}{\mathfrak{k}}\right)\frac{d\phi}{dy}\right] + \frac{d}{dz}\left[\left(1+\frac{4\pi\epsilon}{\mathfrak{k}}\right)\frac{d\phi}{dz}\right] = 0,$$

which, integrated through the given transition-film and treated according to the methods of § 5, gives

$$\left(1+\frac{4\pi\epsilon_1}{\mathfrak{k}}\right)\frac{d\phi_1}{dn} - \left(1+\frac{4\pi\epsilon_0}{\mathfrak{k}}\right)\frac{d\phi_0}{dn} = 0, \quad\ldots\ldots\ldots\ldots(44)$$

the condition sought (cf. also formula (26, VI.)).

The equations of the two preceding articles are known as von Helmholtz's equations of electric and magnetic action at a distance. To obtain the simplest form * of these equations we must eliminate from them all terms that do not refer directly to the immediate neighbourhood of the point, at which the state of the ether is examined; the quantities $\bar{u}'$, $\bar{v}'$, $\bar{w}'$ or U', V', W', integrals extended to all volume-elements of space, are such terms.

* Cf. foot-note, p. 387.

To eliminate these integrals we differentiate equations (16) or (29, XVI.), the second with regard to z and the third to y, and subtract, and we have

$$\frac{d}{dy}\left(\frac{\nu}{\theta}\right)-\frac{d}{dz}\left(\frac{\mu}{\theta}\right)=A\left[\nabla^2\overline{u}'-\frac{d}{dx}\left(\frac{d\overline{u}'}{dx}+\frac{d\overline{v}'}{dy}+\frac{d\overline{w}'}{dz}\right)\right]$$

or, by the relation $\nabla^2\overline{u}'=-4\pi u'$ and formula (23, XVI.),

$$\frac{d}{dy}\left(\frac{\nu}{\theta}\right)-\frac{d}{dz}\left(\frac{\mu}{\theta}\right)=A\left[\mathfrak{e}\frac{d^2\phi}{dx\,dt}-4\pi u'\right].$$

Lastly, by formula (9), we can write this equation as follows:

$$\frac{d}{dy}\left(\frac{\nu}{\theta}\right)-\frac{d}{dz}\left(\frac{\mu}{\theta}\right)=A\left[\mathfrak{e}\frac{d^2\phi}{dx\,dt}-4\pi\left(\frac{\mathfrak{x}'}{\epsilon\kappa}+\frac{d\mathfrak{x}'}{dt}\right)\right];$$

similarly we find

$$\frac{d}{dz}\left(\frac{\lambda}{\theta}\right)-\frac{d}{dx}\left(\frac{\nu}{\theta}\right)=A\left[\mathfrak{e}\frac{d^2\phi}{dy\,dt}-4\pi\left(\frac{\mathfrak{y}'}{\epsilon\kappa}+\frac{d\mathfrak{y}'}{dt}\right)\right]\quad\ldots(45)^*$$

and $$\frac{d}{dx}\left(\frac{\mu}{\theta}\right)-\frac{d}{dy}\left(\frac{\lambda}{\theta}\right)=A\left[\mathfrak{e}\frac{d^2\phi}{dz\,dt}-4\pi\left(\frac{\mathfrak{z}'}{\epsilon\kappa}+\frac{d\mathfrak{z}'}{dt}\right)\right].$$

These equations correspond to Maxwell's fundamental equations (9, II.). To obtain those that replace equations (10, II.) we perform a similar operation on equations (21) or (28, XVI.), and we get

$$\frac{d}{dy}\left(\frac{\mathfrak{z}'}{\epsilon}\right)-\frac{d}{dz}\left(\frac{\mathfrak{y}'}{\epsilon}\right)=A^2\frac{d}{dt}\left(\frac{dV'}{dz}-\frac{dW'}{dy}\right)$$
$$-A\frac{d}{dt}\left[\nabla^2L-\frac{d}{dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)\right]+\frac{d\mathfrak{Z}}{dy}-\frac{d\mathfrak{Y}}{dz}$$

or, upon replacing $\left(\frac{dV'}{dz}-\frac{dW'}{dy}\right)$ by its value from formulae (16a) and ∇^2L by its value $-4\pi\lambda$, the following:

* Cf. von Helmholtz's works, v. 1, p. 622, equations (20e).

$$\frac{d}{dy}\left(\frac{\mathfrak{z}'}{\epsilon}\right)-\frac{d}{dz}\left(\frac{\mathfrak{y}'}{\epsilon}\right)=A\frac{d}{dt}\left(\frac{\lambda}{\theta}+\frac{d\chi}{dx}\right)+4\pi A\frac{d\lambda}{dt}$$

$$+A\frac{d^2}{dt\,dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right)+\frac{d\mathfrak{Z}}{dy}-\frac{d\mathfrak{Y}}{dz}. \quad \ldots(46)$$

We next evaluate the expression

$$\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz};$$

since λ, μ, ν are independent of x, y, z, we can write it explicitly as follows:

$$\int\left[\lambda\frac{d}{dx}\left(\frac{1}{\rho}\right)+\mu\frac{d}{dy}\left(\frac{1}{\rho}\right)+\nu\frac{d}{dz}\left(\frac{1}{\rho}\right)\right]d\tau';$$

by the relations

$$\frac{d}{dx}\left(\frac{1}{\rho}\right)=-\frac{d}{d\xi}\left(\frac{1}{\rho}\right),\quad \frac{d}{dy}\left(\frac{1}{\rho}\right)=-\frac{d}{d\eta}\left(\frac{1}{\rho}\right)$$

and

$$\frac{d}{dz}\left(\frac{1}{\rho}\right)=-\frac{d}{d\zeta}\left(\frac{1}{\rho}\right)$$

and the partial integration of its three component-terms with regard to ξ, η, ζ respectively, we find

$$\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}=\int\left(\frac{d\lambda}{d\xi}+\frac{d\mu}{d\eta}+\frac{d\nu}{d\zeta}\right)\frac{d\tau'}{\rho}$$

or, by formula (32, XVI.),

$$\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}=-\chi. \quad \ldots\ldots\ldots\ldots\ldots(47)$$

Equation (46) can thus be written

$$\frac{d}{dy}\left(\frac{\mathfrak{z}'}{\epsilon}\right)-\frac{d}{dz}\left(\frac{\mathfrak{y}'}{\epsilon}\right)=A\frac{d}{dt}\left(\frac{\lambda}{\theta}\right)+4\pi A\left(\frac{d\lambda}{dt}\right)+\frac{d\mathfrak{Z}}{dy}-\frac{d\mathfrak{Y}}{dz}. \quad \ldots(48)$$

Lastly, if we make the usual assumption that the medium-constant θ is a function of x, y, z only and not of the time t—this condition would in general only be

satisfied for bodies at rest, we can write this equation as follows:

$$\left.\begin{aligned}
&\frac{d}{dy}\left(\frac{\mathfrak{z}'}{\epsilon}\right)-\frac{d}{dz}\left(\frac{\mathfrak{y}'}{\epsilon}\right)=\frac{1+4\pi\theta}{\theta}A\frac{d\lambda}{dt}+\frac{d\mathfrak{Z}}{dy}-\frac{d\mathfrak{Y}}{dz};\\
&\text{the cyclic permutation of the quantities } xyz, \mathfrak{x}'\mathfrak{y}'\mathfrak{z}', \lambda\mu\nu \text{ and } \mathfrak{X}\mathfrak{Y}\mathfrak{Z} \text{ give the two other equations}\\
&\frac{d}{dz}\left(\frac{\mathfrak{x}'}{\epsilon}\right)-\frac{d}{dx}\left(\frac{\mathfrak{z}'}{\epsilon}\right)=\frac{1+4\pi\theta}{\theta}A\frac{d\mu}{dt}+\frac{d\mathfrak{X}}{dz}-\frac{d\mathfrak{Z}}{dx}\\
\text{and }\quad&\frac{d}{dx}\left(\frac{\mathfrak{y}'}{\epsilon}\right)-\frac{d}{dy}\left(\frac{\mathfrak{x}'}{\epsilon}\right)=\frac{1+4\pi\theta}{\theta}A\frac{d\nu}{dt}+\frac{d\mathfrak{Y}}{dx}-\frac{d\mathfrak{X}}{dy}.
\end{aligned}\right\}\ldots(49)^*$$

The six equations (45) and (49) replace Maxwell's fundamental equations (9, II.) and (10, II.). They contain not only the six unknown quantities $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$, λ, μ, ν (P, Q, R, α, β, γ) of Maxwell's equations, but a seventh ϕ. We observe that the function χ does not, however, appear. The former conditional equation (30, XVI.) follows directly from these new equations and cannot therefore be included in the given system. To obtain a seventh independent equation we differentiate equations (21) or (28, XVI.), the first with regard to x, the second to y and the third to z, add, and we find by formula (23, XVI.) the following conditional equation for ϕ:

$$\frac{d}{dx}\left(\frac{\mathfrak{x}'}{\epsilon}-\mathfrak{X}\right)+\frac{d}{dy}\left(\frac{\mathfrak{y}'}{\epsilon}-\mathfrak{Y}\right)+\frac{d}{dz}\left(\frac{\mathfrak{z}'}{\epsilon}-\mathfrak{Z}\right)+\nabla^2\phi$$
$$=A^2\epsilon k\frac{d^2\phi}{dt^2}. \quad\ldots\ldots\ldots\ldots(50)\dagger$$

The conditional equation for χ, namely

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{1}{4\pi}\nabla^2\chi,$$

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* Cf. von Helmholtz's works, v. 1, p. 621, equations (20*c*).

† Cf. the same, v. 1, p. 621, equations (20*d*).

can of course only be regarded as an equation of definition in the given system—for $\mathfrak{e}=0$ the same is also true of the conditional equation (30) for ϕ; χ is here, as in § 37, the potential of the free magnetism, whose density μ is given by the expression

$$\mu=-\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right).$$

Equations (45), (49) and (50) are the simplest form of von Helmholtz's equations of electric and magnetic action —let us distinguish between this new system and the old, where the volume-integrals $\bar{u}', \bar{v}', \bar{w}'$ or U', V', W' appeared, by referring to the former as simply "von Helmholtz's equations"* and to the latter as "von Helmholtz's equations of action at a distance." In our old notation these new equations assume the following form:

Equations (45):

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=\frac{\mathfrak{e}}{\mathfrak{V}}\frac{d^2\phi}{dxdt}-\frac{D-\mathfrak{e}}{\mathfrak{V}}\frac{d}{dt}(P+\mathfrak{X})-\frac{4\pi L}{\mathfrak{V}}(P+\mathfrak{X}), \text{ etc.,}$$

or by formulae (3, XVI.):

$$\left.\begin{aligned}\mathfrak{V}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)&=(D-\mathfrak{e})\frac{dP}{dt}+4\pi L(P+X)-\mathfrak{e}\frac{d^2\phi}{dxdt}\\ \mathfrak{V}\left(\frac{d\gamma}{dx}-\frac{da}{dz}\right)&=(D-\mathfrak{e})\frac{dQ}{dt}+4\pi L(Q+Y)-\mathfrak{e}\frac{d^2\phi}{dydt}\\ \mathfrak{V}\left(\frac{da}{dy}-\frac{d\beta}{dx}\right)&=(D-\mathfrak{e})\frac{dR}{dt}+4\pi L(R+Z)-\mathfrak{e}\frac{d^2\phi}{dzdt}.\end{aligned}\right\}\quad\ldots(51)$$

Equations (49):

$$\frac{dR}{dy}-\frac{dQ}{dz}=\frac{M}{\mathfrak{V}(M-1)}\frac{d}{dt}(M-1)a, \text{ etc.,}$$

* The so-called "von Helmholtz'sche Nahewirkungsgleichungen"; to my knowledge there is no English equivalent for this German word.

or, if the medium-constant M is assumed to be constant with regard to the time:

$$\left.\begin{array}{c}\dfrac{M}{\mathfrak{V}}\dfrac{d\alpha}{dt}=\dfrac{dR}{dy}-\dfrac{dQ}{dz}\\[2ex] \dfrac{M}{\mathfrak{V}}\dfrac{d\beta}{dt}=\dfrac{dP}{dz}-\dfrac{dR}{dx} \quad\text{and}\quad \dfrac{M}{\mathfrak{V}}\dfrac{d\gamma}{dt}=\dfrac{dQ}{dx}-\dfrac{dP}{dy}\end{array}\right\}\ldots(52)$$

—we observe that these last equations are identical with Maxwell's (10, II.). And lastly the conditional equation (50):

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}+\nabla^2\phi=\frac{\epsilon k}{\mathfrak{V}}\frac{d^2\phi}{dt^2}. \;\ldots\ldots\ldots\ldots(53)$$

The equations of definition for ψ (31, XVI.) and (47) assume here the following forms respectively:

$$\frac{d}{dx}(M-1)\alpha+\frac{d}{dy}(M-1)\beta+\frac{d}{dz}(M-1)\gamma=\nabla^2\psi \;\ldots\ldots(54)$$

and

$$\frac{d}{dx}(M-1)\bar{\alpha}+\frac{d}{dy}(M-1)\bar{\beta}+\frac{d}{dz}(M-1)\bar{\gamma}=-4\pi\psi. \;\ldots(55)$$

The above system of equations is identical with Maxwell's only when $\epsilon=0$ (cf. also text p. 356); in this special case they contain quantities that refer in the strictest sense to the state of the ether only in the immediate neighbourhood of the given point. This is no longer true when $\epsilon \lesseqgtr 0$ on account of the appearance of the potential-function ϕ; if, however, we conceive ϕ as a pressure at the given point or as merely an unknown quantity determined by equation (50), we may then regard the given equations as differential equations in the stricter sense, that is, as so called "Nahewirkungsgleichungen" (cf. foot-note to p. 387). For the complete solution of these seven equations we must now evidently know not only the initial ($t=t_0$) values of the unknown

quantities P, Q, R, α, β, γ but also that of $\frac{d\phi}{dt}$: we determine namely $\frac{d\alpha}{dt}$, $\frac{d\beta}{dt}$, $\frac{d\gamma}{dt}$, that is, the rates of change of α, β, γ, from equations (52), then those of P, Q, R from equations (51) and lastly that of $\frac{d\phi}{dt}$ from condition (53). The values of P, Q, R and α, β, γ at any given point and time will thus follow from the values of these six quantities and those of the function ϕ and its derivative $\frac{d\phi}{dt}$ at the given point and at the time directly preceding the given time, and these values will, moreover, depend only on those of the latter quantities in the immediate neighbourhood of the given point. To determine the electric and magnetic forces at *any* point and at *any* time we must evidently know the initial values of P, Q, R, α, β, γ, ϕ, $\frac{d\phi}{dt}$ at *every* point of space.

The values of ϕ and $\frac{d\phi}{dt}$ at any point of space are given by integrals, whose evaluation requires a knowledge of the values of the density of the free electricity ϵ_f' and its rate of change $\frac{d\epsilon_f'}{dt}$ at *every* point of space, since

$$\phi = \frac{1}{\mathfrak{e}}\int\frac{\epsilon_f' d\tau}{\rho} \quad \text{and hence} \quad \frac{d\phi}{dt} = \frac{1}{\mathfrak{e}}\int\frac{\frac{d\epsilon_f'}{dt}d\tau}{\rho}$$

(cf. formulae (28)). It thus follows that the resultant electric or magnetic action at any point is determined not only by the magnitude and direction of the electric and magnetic forces in the immediate neighbourhood of that point but by the source, to which the free electricity at *every* point of space is due, whether to electrostatic accumulations or charges or to electric currents of variable current-strength.

The special form assumed by von Helmholtz's equations

of electricity and magnetism (45), (49) and (50) on the dividing surfaces of adjoining media can be found by assuming the principle of the continuity of transitions and proceeding as in § 5. The second and third of equations (45) evidently give

$$\frac{\mu_1}{\theta_1}-\frac{\mu_0}{\theta_0}=0,\quad \frac{\nu_1}{\theta_1}-\frac{\nu_0}{\theta_0}=0. \quad\ldots\ldots\ldots\ldots\ldots(56)$$

The second and third of equations (49), similarly treated, give

$$\frac{\mathfrak{y}_1'}{\epsilon_1}-\frac{\mathfrak{y}_0'}{\epsilon_0}+\mathfrak{Y}_1-\mathfrak{Y}_0=0,\quad \frac{\mathfrak{z}_1'}{\epsilon_1}-\frac{\mathfrak{z}_0'}{\epsilon_0}+\mathfrak{Z}_1-\mathfrak{Z}_0=0\ldots..(57)$$

To find the equations in λ and $\mathfrak{x}'$ we differentiate equations (49) and (45) respectively, the first with regard to x, the second to y and third to z, add, treat according to the methods of § 5, and we find

$$\frac{d}{dt}\left(\frac{1+4\pi\theta_1}{\theta_1}\lambda_1-\frac{1+4\pi\theta_0}{\theta_0}\lambda_0\right)=0 \quad\ldots\ldots\ldots\ldots(58)$$

and $$\mathfrak{e}\frac{d^2}{dtdx}(\phi_1-\phi_0)-4\pi\left(\frac{\mathfrak{x}_1'}{\epsilon_1\kappa_1}-\frac{\mathfrak{x}_0'}{\epsilon_0\kappa_0}+\frac{d\mathfrak{x}_1'}{dt}-\frac{d\mathfrak{x}_0'}{dt}\right)=0 \quad (59)$$

respectively.

The conditional relation (50) for ϕ treated accordingly assumes the following special form on the dividing-surface of adjoining media:

$$\frac{d}{dx}(\phi_1-\phi_0)=\frac{\mathfrak{x}_0'}{\epsilon_0}-\frac{\mathfrak{x}_1'}{\epsilon_1}+\mathfrak{X}_1-\mathfrak{X}_0;\quad\ldots\ldots\ldots(60)$$

by which formula (59) can be written

$$\frac{\mathfrak{e}}{4\pi}\frac{d}{dt}\left[\left(\frac{1}{\epsilon_1}+\frac{4\pi}{\mathfrak{e}}\right)\mathfrak{x}_1'-\mathfrak{X}_1-\left(\frac{1}{\epsilon_0}+\frac{4\pi}{\mathfrak{e}}\right)\mathfrak{x}_0'+\mathfrak{X}_0\right]$$
$$+\frac{\mathfrak{x}_1'}{\epsilon_1\kappa_1}-\frac{\mathfrak{x}_0'}{\epsilon_0\kappa_0}=0.\ \ldots\ldots(61)$$

These formulae are all referred to a system of coordinates, whose x-axis coincides with the normal to the given dividing-surface.

SECTION XL. EXAMINATION OF THE STATE OF VON HELMHOLTZ'S ETHER IN A HOMOGENEOUS INSULATOR; APPEARANCE OF LONGITUDINAL OSCILLATIONS. ELIMINATION OF THE VARIABLES THAT REPRESENT THE LONGITUDINAL OSCILLATIONS OF A HOMOGENEOUS MEDIUM.

Von Helmholtz's equations of electricity and magnetism are so complicated that they do not admit of a general solution; their various characteristics and the phenomena expressed by them must thus be sought in the examination of special cases. The desired peculiarities may be found most readily upon examining the state of the ether in a homogeneous insulator (ϵ and θ constant, $\kappa=\infty$), within which no electromotive forces reside. Equations (45) assume here the following special form:

$$\left.\begin{aligned}\frac{d\nu}{dy}-\frac{d\mu}{dz}&=A\theta\left[\mathfrak{e}\frac{d^2\phi}{dxdt}-4\pi\frac{dx'}{dt}\right]\\ \frac{d\lambda}{dz}-\frac{d\nu}{dx}&=A\theta\left[\mathfrak{e}\frac{d^2\phi}{dydt}-4\pi\frac{d\mathfrak{y}'}{dt}\right]\\ \frac{d\mu}{dx}-\frac{d\lambda}{dy}&=A\theta\left[\mathfrak{e}\frac{d^2\phi}{dzdt}-4\pi\frac{dz'}{dt}\right];\end{aligned}\right\}\text{.........}(62)$$

equations (49) the following:

$$\left.\begin{aligned}\frac{dz'}{dy}-\frac{d\mathfrak{y}'}{dz}=A\epsilon\frac{1+4\pi\theta}{\theta}\frac{d\lambda}{dt},\quad \frac{dx'}{dz}-\frac{dz'}{dx}=A\epsilon\frac{1+4\pi\theta}{\theta}\frac{d\mu}{dt}\\ \frac{d\mathfrak{y}'}{dx}-\frac{dx'}{dy}=A\epsilon\frac{1+4\pi\theta}{\theta}\frac{d\nu}{dt};\end{aligned}\right\}(63)$$

and condition (50) the following:

$$\frac{1}{\epsilon}\left(\frac{dx'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{dz'}{dz}\right)+\nabla^2\phi=A^2\mathfrak{e}k\frac{d^2\phi}{dt^2}.\text{.........}(64)$$

The differentiation of equations (63), the first with regard to x, the second to y and the third to z, and

their addition give

$$\frac{d}{dt}\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)=0,$$

hence

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\mathrm{f}(x,y,z);$$

that is, this expression is not a function of the time t; hence, if we assume that the given insulator initially contained no free magnetism, none can ever be generated within it, and we shall thus always have

$$\mu_f=-\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)=0 \quad\text{..............(65)}$$

(cf. p. 387).

Lastly, we shall also assume that initially no free electricity resided within the given insulator; no real electricity can then accumulate within it, since the above assumption that $\kappa=\infty$ has excluded all galvanic currents from the system; the free electricity generated at any point will therefore be due alone to electric induction, and we can thus write

$$\phi=\frac{1}{\mathfrak{e}}\int\frac{\epsilon_f' d\tau}{\rho}=\frac{1}{\mathfrak{e}}\int\frac{\epsilon_p' d\tau}{\rho},$$

hence

$$\nabla^2\phi=-\frac{4\pi}{\mathfrak{e}}\epsilon_p'$$

or, since we always have

$$\epsilon_p'=-\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right), \quad\text{..............(66)}$$

$$\nabla^2\phi=\frac{4\pi}{\mathfrak{e}}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right) \quad\text{...............(67)}$$

(cf. also formula (28)).

To find the equations of magnetic action we must eliminate the quantities $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ from the above system; for this purpose we differentiate equations (62), the

second with regard to z and the third to y, subtract, and we find

$$\nabla^2\lambda - \frac{d}{dx}\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right) = -4\pi\theta A\frac{d}{dt}\left(\frac{d\mathfrak{y}'}{dz}-\frac{d\mathfrak{z}'}{dy}\right),$$

which by formulae (63) and (65) reduces to

$$\left.\begin{aligned} &\nabla^2\lambda = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\lambda}{dt^2}; \\ \text{similarly we find } &\nabla^2\mu = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\mu}{dt^2} \\ \text{and } &\nabla^2\nu = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\nu}{dt^2}, \end{aligned}\right\} \quad \ldots\ldots\ldots(68)^*$$

whereby the above condition (65) must always be satisfied.

To obtain the equations of electric action, we differentiate equations (63), the second with regard to z and the third to y, subtract and we get

$$\nabla^2\mathfrak{x}' - \frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right) = A\epsilon\frac{1+4\pi\theta}{\theta}\,\frac{d}{dt}\left(\frac{d\mu}{dz}-\frac{d\nu}{dy}\right)$$

or by equation (62)

$$\nabla^2\mathfrak{x}' - \frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right)$$
$$= 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\mathfrak{x}'}{dt^2} - \epsilon\mathfrak{e}(1+4\pi\theta)A^2\frac{d^3\phi}{dx\,dt^2}. \quad \ldots(69)$$

To eliminate ϕ from this equation we make use of the conditional equation (64); we replace there $\nabla^2\phi$ by its value (67), differentiate with regard to x, and we get

$$\mathfrak{e}A^2\frac{d^3\phi}{dx\,dt^2} = \frac{\frac{1}{\epsilon}+\frac{4\pi}{\mathfrak{e}}}{k}\,\frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right),$$

* Cf. von Helmholtz's works, v. 1, p. 625, equations (21d).

by which equation (69) can be written

$$\nabla^2\mathfrak{x}' = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\mathfrak{x}'}{dt^2} + \left[1 - \frac{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)}{\mathfrak{e}k}\right]\frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right);$$

by cyclic permutation of the quantities x, y, z and $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ we find the two other equations

$$\nabla^2\mathfrak{y}' = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\mathfrak{y}'}{dt^2} + \left[1 - \frac{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)}{\mathfrak{e}k}\right]\frac{d}{dy}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right)$$

and

$$\nabla^2\mathfrak{z}' = 4\pi\epsilon(1+4\pi\theta)A^2\frac{d^2\mathfrak{z}'}{dt^2} + \left[1 - \frac{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)}{\mathfrak{e}k}\right]\frac{d}{dz}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right). \qquad \ldots(70)^*$$

Equations (68) and (70) suffice for the determination of the quantities λ, μ, ν and $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ respectively. They are both special forms of the equations of elasticity; λ, μ, ν correspond to the three component-displacements in the interior of an *incompressible* elastic body, and $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ to those of a solid elastic body, whose density ρ, modulus of elasticity E and ratio of cross-sectional to longitudinal protractions ν are given by the following expressions:

$$\left.\begin{aligned} \rho &= 4\pi A^2 \\ E &= \frac{1}{\epsilon(1+4\pi\theta)}\cdot\frac{3(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)-4\mathfrak{e}k}{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)-\mathfrak{e}k} \\ \nu &= \frac{1}{2}\,\frac{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)-2\mathfrak{e}k}{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)-\mathfrak{e}k}. \end{aligned}\right\} \ldots(71)$$

* Cf. von Helmholtz's works, v. 1, p. 625, equations (21c).

This becomes evident upon writing equations (70) as follows:

$$4\pi A^2\frac{d^2\mathfrak{x}'}{dt^2}-\frac{\mathfrak{e}+4\pi\epsilon}{\epsilon\mathfrak{e}k}\frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right)$$
$$=\frac{1}{\epsilon(1+4\pi\theta)}\left[\nabla^2\mathfrak{x}'-\frac{d}{dx}\left(\frac{d\mathfrak{x}'}{dx}+\frac{d\mathfrak{y}'}{dy}+\frac{d\mathfrak{z}'}{dz}\right)\right], \text{ etc.},$$

and comparing them with the following form of the equations of elasticity:

$$\rho\frac{d^2f}{dt^2}-(\lambda+2\mu)\frac{d}{dx}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)$$
$$=\mu\left[\nabla^2 f-\frac{d}{dx}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)\right], \text{ etc.},$$

where λ and μ are the elastic constants introduced by Lamé (cf. p. 6); for we have

$$\rho=4\pi A^2,\quad \frac{\mathfrak{e}+4\pi\epsilon}{\epsilon\mathfrak{e}k}=\lambda+2\mu,\quad \frac{1}{\epsilon(1+4\pi\theta)}=\mu,$$

hence $\lambda=\dfrac{(1+4\pi\theta)(\mathfrak{e}+4\pi\epsilon)-2\mathfrak{e}k}{\epsilon\mathfrak{e}k(1+4\pi\theta)},\quad \mu=\dfrac{1}{\epsilon(1+4\pi\theta)};$

which, by the following familiar expressions for E and ν, give the above values (71) for these constants:

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\quad \nu=\frac{\lambda}{2(\lambda+\mu)}.$$

As the determination of the general integrals of equations (68) and (70) presents the usual difficulties, we must seek their properties and peculiarities in an examination of their particular integrals. It is evident from the above considerations that the given equations have the form peculiar to all oscillatory motions (cf. also Chapters IV. and V.). Let us examine the different types of plane-waves that can appear in the given medium (insulator); for this purpose we seek the particular integrals that correspond to given types; if,

for example, longitudinal oscillations are included among them, all quantities must be functions of the expression

$$lx+my+nz-at,$$

where l, m, n denote the direction-cosines of propagation of the given oscillations, a their velocity of propagation and t the time. If we lay the x-axis parallel to their direction of propagation, this general condition can be expressed in the following simple form:

$$\left.\begin{array}{l}\lambda=\mathrm{f}(x-at), \quad \mu=\nu=0 \\ \text{for magnetic oscillations, and} \\ \mathfrak{x}'=\mathrm{F}(x-at), \quad \mathfrak{y}'=\mathfrak{z}'=0\end{array}\right\} \quad \ldots\ldots\ldots\ldots(72)$$

for electric oscillations—a denotes the velocity of propagation of the latter; this special form may thus be regarded as the most general condition for the appearance and propagation of longitudinal plane-waves (in any medium).

Similarly, the condition for the appearance and propagation of transverse oscillations in any medium may be written in the special form

$$\left.\begin{array}{l}\lambda=\mu=0, \quad \nu=\mathrm{f}(x-bt) \\ \text{for magnetic oscillations, and} \\ \mathfrak{x}'=\mathfrak{y}'=0, \quad \mathfrak{z}'=\mathrm{F}(x-\beta t)\end{array}\right\} \quad \ldots\ldots\ldots\ldots(73)$$

for electric oscillations, where b and β denote the velocities of propagation of the former and latter respectively; their direction of propagation is here, as above, along the x-axis, whereas that of vibration is parallel to the z-axis.

Let us first examine the magnetic state of the given dielectric. That longitudinal oscillations may be propagated through it,

$$\lambda=\mathrm{f}(x-at), \quad \mu=\nu=0$$

must be particular integrals of equations (68), and they must also satisfy the condition (65). Substituting there

these values, we find

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{d\lambda}{dx}=0, \text{ that is, } \lambda=\text{const.},$$

hence $$0=4\pi\epsilon(1+4\pi\theta)A^2a^2;$$

from which it is evident that the function

$$\lambda=\mathrm{f}(x-at)$$

does not satisfy the given system of equations, that is, that it is not one of its particular integrals; we must thus conclude that no longitudinal (magnetic) oscillations can appear in the given dielectric.

Similarly, we find the following conditions for transverse (magnetic) oscillations:

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{d\nu}{dz}=0, \text{ that is, } \nu=\text{const.}\,(z)=\mathrm{f}(x-bt),$$

hence $$1=4\pi\epsilon(1+4\pi\theta)A^2b^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots(74)$$

which can evidently always be satisfied; conversely, we can thus conclude that $\lambda=\mu=0$ and $\nu=\mathrm{f}(x-bt)$ are particular integrals of the given system and hence that transverse (magnetic) oscillations can be propagated through it; their velocity of propagation b follows directly from the condition (74) and is evidently

$$b=\frac{1}{A}\sqrt{\frac{1}{4\pi\epsilon(1+4\pi\theta)}}; \quad\ldots\ldots\ldots\ldots\ldots\ldots(75)$$

observe that this expression is entirely independent of both the constants $\mathfrak{e}$ and k.

Lastly, let us examine the electric state of our given dielectric. That longitudinal oscillations may appear in it, $\mathfrak{x}'=\mathrm{F}(x-at)$, $\mathfrak{y}'=\mathfrak{z}'=0$ must be particular integrals of equations (70). We replace there $\mathfrak{x}'$ by this function, and we find the following condition for the appearance and propagation of the given oscillations:

$$0=(1+4\pi\theta)\left(4\pi A^2a^2-\frac{\mathfrak{e}+4\pi\epsilon}{\epsilon\mathfrak{e}k}\right);$$

as this condition can evidently always be satisfied, it follows that these values or functions are particular integrals of the given system and hence that longitudinal (electric) oscillations can appear in and be propagated through the given medium; their velocity of propagation is evidently

$$\alpha=\frac{1}{A}\sqrt{\frac{\mathfrak{e}+4\pi\epsilon}{4\pi\epsilon\mathfrak{e}k}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(76)$$

an expression whose value depends on the values assigned the arbitrary constants $\mathfrak{e}$ and k; we postpone its examination to the following article.

Similarly, the condition for the appearance of transverse (electric) oscillations in the given dielectric is

$$\frac{1}{\epsilon}=4\pi(1+4\pi\theta)A^2\beta^2,$$

which gives the following value for their velocity of propagation:

$$\beta=\frac{1}{A}\sqrt{\frac{1}{4\pi\epsilon(1+4\pi\theta)}}. \quad\ldots\ldots\ldots\ldots\ldots(77)$$

We observe that the velocities of propagation of the transverse electric and transverse magnetic oscillations are given by exactly the same expression. Maxwell has already shown that for the special (limiting) case, where $k=0$ and $\epsilon=\theta=\infty$, the electric oscillations are accompanied by magnetic oscillations, whose planes of vibration are at right angles to those of the former, the electric vibrations taking place in the one plane of polarization and the magnetic in the other. That this is also peculiar to von Helmholtz's equations is apparent from an examination of the special case, where electric oscillations, whose planes of vibration are parallel to the z-axis, are propagated along the x-axis; such a disturbance is now characterized by the functions

$$\mathfrak{x}'=\mathfrak{y}'=0, \quad \mathfrak{z}'=\mathrm{F}(x-\beta t);$$

for which equations (63) assume the following special form:

$$\frac{d\lambda}{dt}=\frac{d\nu}{dt}=0, \quad \frac{1+4\pi\theta}{\theta}A\epsilon\frac{d\mu}{dt}-\frac{dF(x-\beta t)}{d(x-\beta t)}=0\,;$$

that is, λ and ν are both constant with regard to the time, whereas μ is a periodic function of t; they represent a periodic (magnetic) disturbance in the xy-coordinate plane at right angles to the x-axis, and along which it is propagated. This (magnetic) disturbance thus takes place at right angles to the given (electric) disturbance and has the same velocity of propagation as the latter, as maintained. On the other hand, it is easy to show that longitudinal (electric) oscillations give rise to no magnetic disturbances whatever.

A peculiarity of the longitudinal (electric) oscillations is that they must always be accompanied by the appearance of free (polarized) electricity, whereas the transverse electric oscillations are characterized by its non-appearance; this follows directly from the substitution of the respective values (72) and (73) for $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ in the expression (66) for the density of the free electricity.

The values found above for the velocities of propagation of the three possible types of ether-oscillations are evidently only their apparent values or those determined by experiment (cf. p. 380); their real values can be obtained by replacing the constant A in the given formulae by its real value (40); we find then

$$\left.\begin{aligned} a&=\mathfrak{A}(\mathfrak{e}+4\pi\epsilon)\sqrt{\frac{1+4\pi\theta}{4\pi\epsilon\mathfrak{e}k}} \\ \text{and}\qquad b&=\beta=\mathfrak{A}\sqrt{\frac{\mathfrak{e}+4\pi\epsilon}{4\pi\epsilon}}. \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(78)$$

Observe that for $\mathfrak{e}=1$ these formulae are identical with those found by von Helmholtz.*

* Cf. works, v. 1, p. 627.

Formulae (78) contain two constants ϵ and $\mathfrak{e}$, which are not entirely independent of each other, as might at first be supposed, but are connected by a given relation, determined by the choice of our system of units (cf. p. 379); if we employ the electrostatic system of units of the medium (air) in question, this relation is given by formula (38); formula (39) is that for the corresponding magnetic system. By the former relation the above formulae (78) can be written in the somewhat simpler form

$$\left.\begin{aligned} a&=\mathfrak{A}\sqrt{\frac{1+4\pi\theta}{4\pi\epsilon k(1-4\pi\epsilon)}} \\ \text{and}\qquad b&=\beta=\mathfrak{A}\sqrt{\frac{1}{4\pi\epsilon}} \end{aligned}\right\} \qquad \text{..........(79)}$$

and by the latter as follows:

$$\left.\begin{aligned} a&=\frac{\mathfrak{A}}{\mathfrak{V}}\sqrt{\frac{1+4\pi\theta}{4\pi\epsilon k(1-4\pi\epsilon\mathfrak{V}^2)}} \\ \text{and}\qquad b&=\beta=\frac{\mathfrak{A}}{\mathfrak{V}}\sqrt{\frac{1}{4\pi\epsilon}}; \end{aligned}\right\} \qquad \text{.........(79}a\text{)}$$

by this elimination of the constant $\mathfrak{e}$ from the above expressions for the velocities of propagation the number of constants they contain has been reduced to three, to two medium-constants ϵ and θ and to one entirely arbitrary constant k.

Lastly, we observe that the function ϕ plays a most important rôle in the above investigations on the state of the ether, its appearance giving rise to longitudinal oscillations; for put $\mathfrak{e}=0$, ϕ will vanish from von Helmholtz's equations, and the latter will reduce to Maxwell's (cf. p. 356), among whose particular integrals such as represent longitudinal oscillatory motion are known to be wanting—this peculiarity of Maxwell's equations can, moreover, be deduced from similar in-

vestigations to those above (cf. also next section). It is possible to transform von Helmholtz's equations for electric and magnetic action in an homogeneous medium, characterized by constant ϵ and κ—these quantities shall, however, be functions of the time—to Maxwell's form by the substitution of new variables for $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$; the given substitution must necessarily be such that the function ϕ will not appear in the resulting equations; this can be accomplished by the substitution

$$\left.\begin{aligned}
\mathfrak{x}' &= \mathfrak{x}'' + \mathfrak{k} e^{-\frac{t}{\epsilon\kappa}} \int_{\infty}^{t} e^{\frac{t}{\epsilon\kappa}} \frac{d^2\phi}{dx\,dt}\,dt \\
\mathfrak{y}' &= \mathfrak{y}'' + \mathfrak{k} e^{-\frac{t}{\epsilon\kappa}} \int_{\infty}^{t} e^{\frac{t}{\epsilon\kappa}} \frac{d^2\phi}{dy\,dt}\,dt \\
\mathfrak{z}' &= \mathfrak{z}'' + \mathfrak{k} e^{-\frac{t}{\epsilon\kappa}} \int_{\infty}^{t} e^{\frac{t}{\epsilon\kappa}} \frac{d^2\phi}{dz\,dt}\,dt,
\end{aligned}\right\} \quad \text{............(80)}$$

which give the following relations:

$$\left.\begin{aligned}
\frac{\mathfrak{x}'}{\epsilon\kappa} + \frac{d\mathfrak{x}'}{dt} &= \frac{\mathfrak{x}''}{\epsilon\kappa} + \frac{d\mathfrak{x}''}{dt} + \frac{\mathfrak{k}}{4\pi}\frac{d^2\phi}{dx\,dt} \\
\frac{\mathfrak{y}'}{\epsilon\kappa} + \frac{d\mathfrak{y}'}{dt} &= \frac{\mathfrak{y}''}{\epsilon\kappa} + \frac{d\mathfrak{y}''}{dt} + \frac{\mathfrak{k}}{4\pi}\frac{d^2\phi}{dy\,dt} \\
\frac{\mathfrak{z}'}{\epsilon\kappa} + \frac{d\mathfrak{z}'}{dt} &= \frac{\mathfrak{z}''}{\epsilon\kappa} + \frac{d\mathfrak{z}''}{dt} + \frac{\mathfrak{k}}{4\pi}\frac{d^2\phi}{dz\,dt}
\end{aligned}\right\} \quad \text{..........(81)}$$

and

$$\left.\begin{aligned}
\frac{d}{dy}\left(\frac{\mathfrak{z}'}{\epsilon}\right) - \frac{d}{dz}\left(\frac{\mathfrak{y}'}{\epsilon}\right) &= \frac{d}{dy}\left(\frac{\mathfrak{z}''}{\epsilon}\right) - \frac{d}{dz}\left(\frac{\mathfrak{y}''}{\epsilon}\right) \\
\frac{d}{dz}\left(\frac{\mathfrak{x}'}{\epsilon}\right) - \frac{d}{dx}\left(\frac{\mathfrak{z}'}{\epsilon}\right) &= \frac{d}{dz}\left(\frac{\mathfrak{x}''}{\epsilon}\right) - \frac{d}{dx}\left(\frac{\mathfrak{z}''}{\epsilon}\right) \\
\frac{d}{dx}\left(\frac{\mathfrak{y}'}{\epsilon}\right) - \frac{d}{dy}\left(\frac{\mathfrak{x}'}{\epsilon}\right) &= \frac{d}{dx}\left(\frac{\mathfrak{y}''}{\epsilon}\right) - \frac{d}{dy}\left(\frac{\mathfrak{x}''}{\epsilon}\right);
\end{aligned}\right\} \quad \text{.......(82)}$$

by the former of these relations von Helmholtz's equations (45) reduce to

$$\left.\begin{aligned}4\pi A\frac{d\mathfrak{x}''}{dt}&=\frac{d}{dz}\left(\frac{\mu}{\theta}\right)-\frac{d}{dy}\left(\frac{\nu}{\theta}\right)-\frac{4\pi A}{\epsilon\kappa}\mathfrak{x}''\\4\pi A\frac{d\mathfrak{y}''}{dt}&=\frac{d}{dx}\left(\frac{\nu}{\theta}\right)-\frac{d}{dz}\left(\frac{\lambda}{\theta}\right)-\frac{4\pi A}{\epsilon\kappa}\mathfrak{y}''\\4\pi A\frac{d\mathfrak{z}''}{dt}&=\frac{d}{dy}\left(\frac{\lambda}{\theta}\right)-\frac{d}{dx}\left(\frac{\mu}{\theta}\right)-\frac{4\pi A}{\epsilon\kappa}\mathfrak{z}''\end{aligned}\right\}\ldots\ldots(83)$$

and by the latter his equations (49) to

$$\left.\begin{aligned}\frac{1+4\pi\theta}{\theta}A\frac{d\lambda}{dt}&=\frac{d}{dy}\left(\frac{\mathfrak{z}''}{\epsilon}\right)-\frac{d}{dz}\left(\frac{\mathfrak{y}''}{\epsilon}\right)+\frac{d\mathfrak{Y}}{dz}-\frac{d\mathfrak{Z}}{dy}\\\frac{1+4\pi\theta}{\theta}A\frac{d\mu}{dt}&=\frac{d}{dz}\left(\frac{\mathfrak{x}''}{\epsilon}\right)-\frac{d}{dx}\left(\frac{\mathfrak{z}''}{\epsilon}\right)+\frac{d\mathfrak{Z}}{dx}-\frac{d\mathfrak{X}}{dz}\\\frac{1+4\pi\theta}{\theta}A\frac{d\nu}{dt}&=\frac{d}{dx}\left(\frac{\mathfrak{y}''}{\epsilon}\right)-\frac{d}{dy}\left(\frac{\mathfrak{x}''}{\epsilon}\right)+\frac{d\mathfrak{X}}{dy}-\frac{d\mathfrak{Y}}{dx}.\end{aligned}\right\}\ldots(84)$$

The similarity between these and Maxwell's fundamental equations (9, II.) and (10, II.) respectively becomes apparent upon recalling to memory the form in which von Helmholtz has introduced the external electromotive forces (cf. formulae (3, XVI.)). As ϕ no longer appears in these equations, von Helmholtz's conditional equation (50) for this seventh variable becomes superfluous here. We can thus conclude that the behaviour of our new variables $\mathfrak{x}''$, $\mathfrak{y}''$, $\mathfrak{z}''$ in the given medium is exactly analogous to that of Maxwell's quantities P, Q, R, hence that they represent only its transverse oscillations and that these, similarly to Maxwell's waves, advance entirely independent of its longitudinal ones. We should not, however, wrongly suppose that our new system of equations in $\mathfrak{x}''$, $\mathfrak{y}''$, $\mathfrak{z}''$ replaces von Helmholtz's proper, or that these new variables define the total resultant state of the given medium; they constitute indeed only the

one set of factors or components, that define its composite state, and have been obtained by the elimination of its others—the function ϕ. Lastly, we should observe that the desired elimination is only possible, when ϵ and κ are constant with regard to x, y, z; conversely, we must conclude that the transverse oscillations of a non-homogeneous medium or those of a homogeneous medium, upon entering another medium, cannot be independent on its longitudinal ones.

If we were ever obliged to abandon Maxwell's theory on account of its shortcomings and to accept a more general one, as von Helmholtz's, the above and similar substitutions would evidently be of great service in the examination of the transverse oscillations of all homogeneous media.

SECTION XLI.—EXAMINATION OF THE EXPRESSION FOR THE VELOCITY OF PROPAGATION OF THE LONGITUDINAL ETHER-OSCILLATIONS. ASTRONOMICAL RESEARCHES TO FIND A VELOCITY OF PROPAGATION FOR GRAVITATION: DIRECT METHODS; DYNAMICAL ANALOGIES; SEELIGER'S PROOF OF INVALIDITY OF NEWTON'S LAW. THE CATHODE AND RÖNTGEN RAYS; JAUMANN'S MODIFICATION OF MAXWELL'S THEORY.

Before we shall be able to form any idea of the possible numerical values that the expression for the velocity of propagation of the longitudinal oscillations of the ether may assume, we must know the value of the medium-constant ϵ; the latter depends of course on the system of units employed. The value of ϵ can be determined from those of the observed ($\mathfrak{A}$) and the real (β) velocities of propagation of the transverse ether-waves, that is, from those of the velocity of propagation of the electric waves (determined by experiment) and of that

of light; suppose we take the velocity of propagation of the former as 1% smaller than that of the latter—such a variation could surely be regarded as a maximum—we should then have by formulae (79)

$$\beta - \mathfrak{A} = \beta(1 - \sqrt{4\pi\epsilon}) = \frac{\beta}{100},$$

which gives $$\epsilon = \frac{1}{4\pi}\left(\frac{99}{100}\right)^2 = 0{\cdot}078\,;$$

in our other system of units, that to which formulae (79*a*) refer, ϵ would assume the value

$$\epsilon = \frac{0{\cdot}078}{\mathfrak{B}^2}.$$

We should thus find the following value for a in both systems:

$$a = \frac{50}{7}\sqrt{\frac{1+4\pi\theta}{k}}\mathfrak{A}.$$

Since the constant of magnetic induction θ is always positive, we could thus write

$$a > \frac{50}{7}\sqrt{\frac{1}{k}}\mathfrak{A} > 7\mathfrak{A}\sqrt{\frac{1}{k}}. \qquad (85)$$

For $k = -1$, the value assigned this arbitrary constant by Weber, a becomes imaginary; it follows, therefore, that, in accepting Weber's theory of electricity and magnetism, we are excluding all longitudinal oscillations from the ether. This shortcoming in his theory is, moreover, to be attributed to a serious inconsistency in Weber's equations, when applied to motions in any but closed circuits, as first pointed out by von Helmholtz;* he shows, namely, that, for all negative values of k, the

* "Ueber die Gesetze der inconstanten electrischen Ströme in körperlich ausgedehnten Leitern." *Wissenschaftliche Abhandlungen*, v. 1., pp. 537-544.

total energy of any system, in which external electromotive forces have ceased to act, if once negative, will assume larger and larger negative values, until it finally becomes (negatively) infinitely large; and hence that Weber's equations of electric action must lead in the given case to an unstable equilibrium; whereas for all positive values of k, including $k=0$ (Maxwell's potential), the resulting equilibrium remains stable.

For $k=0$, the value corresponding to Maxwell's potential (cf. end of § 37), a becomes infinitely large, that is, this special form of von Helmholtz's equations also does not admit the appearance of longitudinal oscillations.

For $k=1$, Neumann's potential, the above formula (85) would give

$$a > 7\mathfrak{A},$$

that is, the minimum velocity of propagation of the longitudinal oscillations would be here 7 times that of light.

It is evident from the above that as the observed velocity of propagation of the transverse electric oscillations approaches that of light a increases. We have already seen in § 17 that, in order to effect an agreement (in our concrete representation) between electrostatic phenomena and the Hertzian oscillations, we must choose the constant of electric polarization ϵ in air very small in comparison to unity; this corresponds to the supposition that the observed and the real velocities of propagation of the transverse ether-oscillations are approximately the same, whereby all appreciable variations would have to be ascribed to errors of observation, and hence to the assumption of an enormous velocity of propagation for its longitudinal ones. Just such an enormous velocity of propagation is required of the gravitation-waves, if we are to hope to explain the perturbations in the orbits of our planets by the fact that gravitation does not act directly at a distance, but that it requires time for its propagation, that is, that it is a so-called indirect action or, more correctly, a phenomenon, whose explanation is

alone to be sought in the state of some medium pervading entire space. If we regard the luminiferous ether, as defined by von Helmholtz's equations, as the given medium or transmitter of so-called gravitating action, we are then able, on the one hand, to interpret its longitudinal oscillations, namely, as gravitation-waves propagated through space with the given enormous velocity, and, on the other hand, to form some conception of the mysterious force of gravitation itself, for we can then conceive it as a medium-stress arising from a certain type of ether-oscillations, its longitudinal ones, that pervade entire space.

The various attempts made to explain the secular perturbations in the orbits of our planets and our moon, by assuming a given velocity of propagation for gravitation, and conversely to determine this velocity from these calculations, not only throw light on the ensuing considerations but are of so universal scientific interest that a brief examination of their results can hardly be out of place here. These investigations date back as far as Laplace*; he assigns the gravitating force a finite velocity of propagation and conceives that the given planet is subjected to a disturbing force, whose direction of action is not along its radius vector (from the sun) but deviates slightly from it and whose magnitude is to that of the sun's attraction as the velocity of propagation of the latter is to that of the planet itself. Dr. S. Oppenheim† has recently shown that the velocity of propagation of gravitation would have to be taken here 12,000,000 times

* *Mécanique céleste*: book x., chap. vii., "Sur les altérations que le mouvement des planètes et des comètes peut éprouver par la résistance des milieux qu'elles traversent et par la transmission successive de la pesanteur"; book XVI., chap. iv., "Sur la diminution de l'attraction par l'interposition des corps"; and book IV., chap. xvii., "Exposition du système du monde."

† "Zur Frage nach der Fortpflanzungsgeschwindigkeit der Gravitation," § 1, *aus dem Jahresberichte über das k. k. akademische Gymnasium in Wien für* 1894-95.

that of light, to account for the small variation of one second per century in the mean longitude of our earth, and 12,400,000 times that velocity, to explain the variation of 6 seconds (per century) in that of our moon, results that agree most strikingly with each other. Professors Lehmann-Filhés* and Heppergert† have, on the other hand, attempted to solve the problem directly; they assume namely, that Newton's law of action, which is, strictly speaking, valid only for bodies at rest, also holds for moving bodies, but with the slight modification that the distance (r) between the attracting centres at the given time t_0 be replaced by their distance apart at the time $t_0 - \frac{r}{c}$, where c denotes the velocity of propagation of gravitation; this corresponds to the assumption that gravitation like light is propagated radially from given centres and that the gravitation waves advance, like those of light, with a constant uniform velocity. In developing the equations of motion for such a system, two serious difficulties are encountered: in the first place, the disturbing forces cannot be represented as the partial derivatives with regard to the coordinates of any given function, so that the principle of the conservation of energy cannot remain valid, and secondly, the final expression for the given perturbation contains terms that are functions of the velocity of translation of our solar system through space, the determination of whose value is still a problem of the greatest uncertainty—the

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* "Ueber die Bewegung eines Planeten unter der Annahme einer sich nicht momentan fortpflanzenden Schwerkraft": *Astronomische Nachrichten*, v. 110, 1884; and "Ueber die Säcularstörung der Länge des Mondes unter der Annahme einer sich nicht momentan fortpflanzenden Schwerkraft": *Sitzungsberischte der math-physikalischen Classe der k. bayer. Akademie der Wissenschaften*, v. 25, part iii., 1895.

† "Ueber die Fortpflanzungsgeschwindigkeit der Gravitation": *Sitzungsberichte der kais. Akademie der Wissenschaften in Wein, math-naturwissenschaftliche Classe*, v. 97, part ii. *a*, March 1888.

methods at present in vogue give values ranging from 1·6 to 30·0 km. per second. We can thus lay little stress on the different values found for the velocity of propagation of gravitation according to this method; we may observe, however, that they all tend towards a velocity, whose dimensions are those of the velocities found by the previous method, rather than one whose dimensions are those of the velocity of light.

Lastly, many attempts have been made in recent years to explain the above variations in the elements of the orbits of our planets by assuming laws deduced from dynamical analogies in place of Newton's. Electrodynamics alone has furnished these analogies; they have been obtained by adding to Newton's law correction-terms similar to those that it has been found necessary to add to Coulomb's, in order that the latter in its modified form might give the correct values for the ponderable forces; these correction-terms represent the forces, to which the motions of the bodies in question give rise. The determination of the electrodynamical correction-terms is still a problem of the future; the ones most commonly known and forming the basis for further investigation are, however, those given by Gauss, W. Weber, Riemann and Clausius. Gauss first called attention not only to the electrodynamical problem,* but to the possible analogy between it and that of gravitation; whereas the electrodynamical potentials assumed by the three later scientists have furnished the material for extensive investigations in the analogous problem of gravitation. The assumption of the given analogy between gravitation and electrodynamics is of course equivalent to the supposition that both gravitating and electric actions are similarly propagated, and, moreover, that the mechanism which propagates the one is identical to that which propagates the other.

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* Gauss' works, vol. 5, Nachlass, p. 627: "Aus einem Briefe von Gauss an W. Weber"; 1845.

Weber's potential gives the following analogous expression for the (ponderable) energy between two material particles:

$$P=\frac{k^2 m_0 m_1}{r}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right], \quad\ldots\ldots\ldots\ldots\ldots(86)$$

where k is a constant, m_0 and m_1 the masses of the two particles (planets), r their distance apart and c the velocity of propagation of the given action. Riemann's potential gives the following expression for this energy:

$$P=\frac{k^2(m_0+m_1)}{r}\left\{1-\frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]\right\}. \ldots(87)$$

The numerous investigations* founded on the assumption of the former potential show that the velocity c must be taken smaller than that of light, namely $c=0{\cdot}57\mathfrak{B}$, to give the observed variation of $41''{\cdot}25$ per century in the longitude of the perihelion of Mercury; those† assuming Riemann's potential have found approximately the same value for this velocity ($c=0{\cdot}81\mathfrak{B}$).

Lastly, in assuming Clausius' potential,‡ namely

$$P=\frac{k^2(m_0+m_1)}{r}\left[1+\frac{1}{c^2}\left(\frac{d\xi_0}{dt}\frac{d\xi_1}{dt}+\frac{d\eta_0}{dt}\frac{d\eta_1}{dt}+\frac{d\zeta_0}{dt}\frac{d\zeta_1}{dt}\right)\right], \ (88)$$

* *F. Tisseraud*: "Sur le mouvement des planètes autour du soleil d'aprés la loi électrodynamique de Weber"; *Comptes Rendus*, vol. 75, 1872, p. 760.

Holzmüller: "Ueber die Anwendung der Jacobi-Hamilton'schen Methode auf den Fall der Anziehung nach dem electrodynamischen Gesetze von Weber"; *Schlömilchs Zeitschrift*, 1870.

H. Servus: "Untersuchungen über die Bahn und die Störungen der Himmelskörper mit Zugrundelegung des Weber'schen electrodynamischen Gesetzes"; *Inauguraldissertation*, Halle, 1885.

S. Oppenheim: "Zur Frage nach der Fortpflanzungsgeschwindigkeit der Gravitation"; *Separatabdruck aus dem Jahresberichte über das k. k. Akademische Gymnasium in Wien;* 1894-95. § IV.

† *Oskar Liman*: "Die Bewegung zweier materieller Punkte unter Zugrundelegung des Riemann'schen electrodynamischen Gesetzes"; *Inauguraldissertation*, Halle, 1886.

S. Oppenheim: same as above, § V.

‡ Cf. Poggendorf's *Annalen*, vol. 156.

where ξ_0, η_0, ζ_0 and ξ_1, η_1, ζ_1 denote the absolute coordinates of the two planets m_0 and m_1, we naturally obtain an expression for c that contains these coordinates; this, of course, renders an exact numerical determination of the given velocity impossible; its most probable values are, however, somewhat smaller than that for the velocity of light.

We observe that, in all the above investigations founded on the existence of a potential similar to those employed in electro-dynamics, the velocity of propagation of the gravitation-waves must be taken as small as or even smaller than that of light to explain the observed variations in the orbit elements, and, moreover, that these velocities of propagation are of entirely different dimensions from those already found by the first two methods; it seems to me that the values determined by the latter must be given the preference, if there be any, and for the following reasons: There are only two possibilities; since the very conception of indirect action necessarily implies the existence of an intervening medium or ether, gravitation must be the action of either its longitudinal or its transverse vibrations. The values for c found by the last method, the electrodynamical analogy, are now entirely inconsistent with the explanation of gravitation as due to the action of longitudinal waves, first, because all longitudinal vibrations, whose velocity of propagation it has been possible to determine experimentally, as those of gelatin, glass, etc., have been found to be propagated with a velocity appreciably greater than that of their transverse ones, and secondly, because the above electrodynamical analogy proves upon closer examination not to be so feasible as at first sight; for what analogy really exists between the longitudinal and the transverse disturbances of a medium? or what right has one to suppose, because the action arising from the (relative) motion of two conductors charged with electricity, whose presence we assume to be manifested by certain transverse displace-

ments of an ether, can be represented approximately by various analytical expressions, that the action due to the motions of the planets—if the preceding be taken for granted, this action must then be attributed to the presence of its longitudinal waves—should be given by analogous expressions? On the other hand would not this dissimilarity rather render any such analogy impossible? Our alternative would be to assume that gravitation is the action of only another type of transverse vibrations of the given ether; the above electrodynamical analogy might then hold, as far as any reasons that have already been mentioned are concerned. Let us next examine this supposition.

We first observe that all laws of action are only approximate; they are, in fact, merely convenient dynamical analogies, that represent approximately the observed actions, and not natural laws. The simplest such are Coulomb's and Newton's laws; the former cannot, however, be deduced from either Maxwell's or von Helmholtz's equations of electricity and magnetism; Coulomb's law has, in fact, only been regarded as a feature of our so-called concrete representation (cf. §§ 16 and 39), by means of which we have been enabled to represent (approximately) the observed actions. The same would naturally be true of Newton's law also, provided we granted the existence of an intervening medium. On the other hand, Prof. Seeliger* has recently shown that Newton's law of gravitation cannot be a natural law; he, of course, recognizes the fact that it suffices for the solution of most of the problems in our planetary system, although it undoubtedly requires some slight correction to explain certain variations, observed in special or limiting cases. We may safely maintain that this point of view is also taken by most astronomers and scientists; but the first direct proof of

* "Ueber das Newton'sche Gravitationsgesetz"; *Astronomische Nachrichten*, vol. 137, 1894.

the invalidity of Newton's law or statement of the many inconsistencies embodied in it is that given by Prof. Seeliger.

Seeliger's proof of the invalidity of Newton's law is founded on the inconsistencies, which he has been able to deduce from well-known theorems of the potential, when extended to the attraction of masses at infinite distances apart. He shows that the given inconsistencies can be eliminated by the assumption of various other expressions in place of Newton's; the difficulty, however, is in making a choice among them, for it naturally involves that of deciding upon one of the many feasible assumptions or analogies that suggest themselves. As example of such an analogy Prof. Seeliger takes that of the absorption of light and thus replaces Newton's law by the following

$$k^2mm'\frac{e^{-\lambda r}}{r^2}, \qquad (89)$$

where λ is to be chosen so small that this expression reduces approximately to Newton's for all but certain limiting cases.

It is evident from the above that the number of expressions that would fulfil the required conditions is so large and that the values for the velocity c of propagation of gravitating action, provided this variable enter as factor, are thus so different, that no weight whatever can be laid on the dynamical analogies, either the physical or the purely mechanical ones. In consideration of these facts it seems to me that we are perfectly justified in discarding the last method and also the many values for the velocity c of propagation found by means of it, at least, until we have some stronger arguments for a choice among the many possible analogies than our present knowledge offers; and in the meantime in retaining the value found for this velocity c by the direct methods, the first and second.

The great variation found in the values for the velocity

of propagation of gravitating action by the above methods might at first sight lead one to conclude that the given secular variations in the orbits of our planets could not be explained by the assumption that gravitating action requires time for its propagation and thus to abandon entirely the only plausible assumption of indirect action or a medium; but, on the other hand, are not the very facts, that all attempts to find laws of both gravitating and electric actions are really only those to obtain good dynamical analogies to reproduce these actions, that these analogies are so numerous and, lastly, that the natural laws we often presume we are seeking do not perhaps exist at all, sufficient proof that a medium or ether must exist? Moreover, do not the mysteries of the electric mechanism also justify these conclusions (cf. also below)?

If we grant the preceding, we are then reduced to the one possibility, namely that gravitation is the action of longitudinal ether-waves whose velocity of propagation is several millions (12,000,000) times that of light. The action imparted to any body might then be conceived as due to the medium-stress in its immediate neighbourhood, arising from the resultant action of the longitudinal waves emitted by the surrounding bodies; this action must undoubtedly be of a very complicated nature, but to what degree it depends on the distances between the given bodies, the constitution of the intervening medium, changes in the latter arising from motions of the former, etc., is a problem that must be deferred to futurity.

To what category of ether-oscillations the cathode and Röntgen (X) rays belong, is a question that cannot be answered, until their many properties and peculiarities have been more thoroughly established. In the meantime, we must be satisfied with confining our investigations to negative results only; we shall find, namely, upon more careful examination that there are certain theories offered as explanations of these rays that

must be rejected, and we may thus hope to reduce the number of current theories or explanations to a minimum.

Upon the discovery of the Röntgen rays the general tendency was to regard them as longitudinal oscillations of an ether; it was universally recognized that they could not be those of a Maxwell's ether, for his theory does not admit of the appearance or propagation of longitudinal oscillations of any type (cf. pp. 400-401). Although the ether, as defined by von Helmholtz's more general equations, can be the transmitter of longitudinal waves and, conversely, the cathode and Röntgen rays might, from a purely theoretical standpoint, be regarded as a certain type of these waves, there are many empirical reasons for refuting such a supposition; these are to be found in the strong resemblance between the longitudinal waves of von Helmholtz's ether and those of sound and, on the other hand, in the marked contrast between their several peculiarities and those of the cathode and Röntgen rays. For these reasons alone the possibility of explaining the cathode or Röntgen rays as the longitudinal oscillations of von Helmholtz's ether received little attention. The only alternative here was thus to regard the Röntgen rays as the longitudinal oscillations of a slightly modified Maxwell's ether, that would permit the appearance and propagation of such oscillations; among others Jaumann's theory for the explanation of the cathode rays as longitudinal waves was taken as the one sought for that of the Röntgen, and the latter were regarded as a given type of these oscillations. This and similar such explanations were so readily accepted by the majority, that it does not seem out of place to examine them briefly here; as, what is true of the one theory, applies, in general, to them all, let us take, as example, the modification introduced by Jaumann into Maxwell's equations. We should, however, realize that it is a matter of no little moment to reject such equations as Maxwell's, which have stood the test of years, and

accept new ones in their place, merely for the purpose of explaining a certain type of rays.

Jaumann's theory* is founded on the assumption that Maxwell's medium-constants D and M do not remain constant in rarified air or a vacuum, but that they vary with the time; in other media, as the atmosphere, they are supposed to remain constant or, at least, to vary so little during the passage of electric disturbances that their variations may be entirely neglected. Jaumann's modified form of Maxwell's equations for rarified gases ($L=0$) thus becomes

$$\left.\begin{aligned} \frac{d}{dt}(DP) &= \mathfrak{V}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right) \\ \frac{d}{dt}(DQ) &= \mathfrak{V}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right) \\ \text{and}\qquad \frac{d}{dt}(DR) &= \mathfrak{V}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(90)$$

$$\left.\begin{aligned} \text{and}\qquad \frac{d}{dt}(M\alpha) &= \mathfrak{V}\left(\frac{dR}{dy}-\frac{dQ}{dz}\right) \\ \frac{d}{dt}(M\beta) &= \mathfrak{V}\left(\frac{dP}{dz}-\frac{dR}{dx}\right) \\ \text{and}\qquad \frac{d}{dt}(M\gamma) &= \mathfrak{V}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(91)$$

(cf. equations (9, II.) and (10, II.)); we observe that these equations are not linear like Maxwell's. The six equations (90) and (91) contain eight variables, P, Q, R, α, β, γ, D, M; to supply the two wanting equations, Jaumann assumes that D and M are given by the differential equations

$$\frac{a_0}{\eta_0-1}\frac{dD}{dt}=\frac{d}{dx}(D_0P)+\frac{d}{dy}(D_0Q)+\frac{d}{dz}(D_0R) \quad \ldots\ldots(92)$$

*"Longitudinales Licht," *Sitzungsberichte der k. Akademie der Wissenschaften, math-naturwissenschaftliche Classe*, vol. 104, 1895, pp. 747-792.

and

$$\frac{a_0}{\eta_0 - 1} \cdot \frac{\eta_0}{\nu_0} \frac{dM}{dt} = \frac{d}{dx}(D_0 P) + \frac{d}{dy}(D_0 Q) + \frac{d}{dz}(D_0 R), \ldots\ldots (93)$$

where D_0 denotes the value of D before the passage of the disturbance and a_0 a positive reduction-factor, whose dimensions are those of an electric force divided by a velocity. η_0 and ν_0 are two new medium-constants; the former has the property that for $\eta_0 = 0$ or $\eta_0 = 1$ Jaumann's equations reduce to Maxwell's; the latter is a function of the density of the medium, it has the value unity for dense media but vanishes for a vacuum. These equations (92) and (93) are of course to be regarded as equations of definition for the quantities D and M, which characterize the electric and magnetic properties of the given medium; the assumption of their validity takes, of course, for granted a complete knowledge of the behaviour of quantities, about which we really know nothing whatever; this alone seems sufficient reason for laying no weight either on their validity or on the peculiar properties Jaumann has been able to deduce from his system of equations, for all such properties are, of course, to be attributed entirely to the equations assumed as definitions for D and M. Particular integrals of Jaumann's equations, that represent longitudinal waves, are given and discussed in § 4 of his paper.

Apart from the above objection to the validity of equations (92) and (93), there is an inconsistency of a purely empirical nature embodied in the assumption of equations (90) and (91) in reference to the cathode rays; Jaumann supposes namely that the rarified air within the tube, when subjected to the given powerful electric disturbances, undergoes changes in its constitution, defined by equations (92) and (93), which give rise to the formation and propagation of longitudinal waves, among them the cathode rays; but, on the other hand, that the air without the tube is so dense, compared to that within it, that similar changes in the constitution of the former

may be entirely neglected. If this be the case, how are we to explain by Jaumann's theory the passage of the cathode rays through the sides of the tube and their propagation thence into space? Or are we perhaps to assume that the denser media also undergo appreciable changes in their constitution? This only alternative would demand the substitution of Jaumann's equations in place of Maxwell's for all media; as such a step would, however, involve the not only laborious but vain task of confirming Jaumann's equations for all phenomena, practically all of which have already been most satisfactorily explained by Maxwell's theory, we are surely gaining nothing in taking it. We observe, moreover, that the difference in density between the air within and without the tube is, strictly speaking, one only between their respective ethers, and hence, upon taking this into consideration, that the above difficulties only become the more formidable.

We must conclude from the above that the cathode and Röntgen rays cannot be satisfactorily explained as the longitudinal waves of any ether yet familiar to us. But, on the other hand, we can maintain that there is no theoretical reason to prevent us from conceiving them as a certain type of transverse oscillations of either a Maxwell's or a von Helmholtz's ether. In fact, do not recent experiments tend to show—as Stokes suggests—that the Röntgen rays belong to a part of the spectrum beyond the ultra violet? and, moreover, would not the extremely short wave-lengths of such rays offer a satisfactory explanation for their peculiar property of being able to penetrate bodies that are opaque to other kinds of light? The answers to these and to similar questions must, however, be deferred till the many properties and peculiarities of these rays have been more thoroughly established.

CHAPTER XVIII.

SECTION XLII. MAXWELL'S EQUATIONS OF ELECTRIC AND MAGNETIC ACTION FOR MOVING BODIES; BEHAVIOUR OF THE REAL ELECTRICITY AND MAGNETISM; INDUCTION.

THE electric and magnetic phenomena due to the relative motion of given bodies, magnets, wires carrying electric currents, etc., were first observed by Faraday. Maxwell was, however, the first to establish equations for moving bodies; they are more general than those for bodies at rest, since the medium-constants are assumed to remain constant with regard to the time in the derivation of the latter, whereas this assumption must evidently be abandoned for bodies in motion. The real importance of such equations becomes apparent, when we realize that (Maxwell's) equations for bodies at rest express only given stationary but not the transitory states of the ether, and that in excluding the latter from investigation we are allowing all possible variations in its state, such, for example, as might give rise to the creation or destruction of electricity or magnetism, to escape our notice. Still more general equations than Maxwell's are those developed by von Helmholtz; their derivation is similar to his in every respect, the equations themselves are only heavier and more complicated; for this reason we shall retain Maxwell's fundamental equations in the following.

The fundamental principle in the derivation of all

equations of electric and magnetic action for bodies in motion is Faraday's conception of the lines of force or induction. To find the equations that replace Maxwell's (10, II.), we make use of the lines of magnetic induction; denoting the vector of the magnetic induction at any point by V, we have

$$\left.\begin{array}{c} V=M\sqrt{a^2+\beta^2+\gamma^2}, \quad \cos(V, x)=\dfrac{Ma}{V}, \\ \cos(V, y)=\dfrac{M\beta}{V}, \quad \cos(V, z)=\dfrac{M\gamma}{V}. \end{array}\right\} \quad \ldots\ldots(1)$$

For air ($M=1$) the lines of magnetic induction become identical to those of magnetic force. Next let $Vdo=n$, where n is a chosen integer, denote the number of lines of magnetic induction drawn through any surface-element do at right angles to the vector V. The number of lines n' that pass through any surface-element df will then be

$$n'=Vdf\cos(V, N)=Vdo', \quad \ldots\ldots\ldots\ldots\ldots(2)$$

where N is the normal to the given surface and do' the projection of this surface on do. If $\angle(V, N)>90°$, $\cos(V, N)$ will be negative and hence n' also. If F denotes the number of lines of induction that pass through any surface-element df parallel to the yz-coordinate-plane, formulae (1) and (2) will then give

$$F=Madf;$$

similarly we find

$$G=M\beta dg, \quad H=M\gamma dh.$$

The variation in the number of lines of induction that pass through the surface-element df (at rest) during the period δt will thus be

$$\delta F=M\delta adf=M\frac{da}{dt}\delta t\,df$$

or, by formulae (10, II.),

$$\left.\begin{aligned} &\delta F = \mathfrak{V}\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) df\,\delta t; \\ &\text{similarly we find} \\ &\delta G = \mathfrak{V}\left(\frac{dP}{dz} - \frac{dR}{dx}\right) dg\,\delta t, \quad \delta H = \mathfrak{V}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dh\,\delta t. \end{aligned}\right\} \quad \ldots (3)$$

We next make the assumption of Faraday, von Helmholtz and Hertz that moving bodies carry their lines of induction along with them through space—the consequences of such an assumption are at least consistent with experience; the above expressions (3) for $\delta F, \delta G, \delta H$ will then hold for moving bodies. Hereby the relative position of the surface-element df to the coordinate-axes will however have changed and the element itself also have undergone slight deformations.

Other expressions for δF, δG, δH can now be found by determining the variation in the number of lines of magnetic induction that pass through df, dg, dh respectively, as these surfaces are carried along through space, arising from the variation in the value of the vector V from point to point. Take the initial and next succeeding positions of the surface-element df; if we characterize the latter position by suffixing a dash (′) to the given quantities, we have

$$F' = M'\alpha' df_1',$$

$$G' = M'\beta' df_2' \quad \text{and} \quad H' = M'\gamma' df_3',$$

where df_1', df_2', df_3' denote the projections of df' on the three coordinate-planes. We have, therefore,

$$\begin{aligned} \delta F &= F' + G' + H' - F \\ &= M'\alpha' df_1' + M'\beta' df_2' + M'\gamma' df_3' - M\alpha df. \quad \ldots\ldots (4) \end{aligned}$$

The quantities M, α, β, γ will, in general, be functions of x, y, z, t. To determine M', α', β', γ' we must thus know the velocity of translation of the given body

(surface-element); if we denote its component velocities by λ, μ, ν and write

$$Ma=\phi(x, y, z, t),$$

we have

$$M'a'=\phi(t+\delta t, x+\lambda\delta t, y+\mu\delta t, z+\nu\delta t)$$

$$=Ma+\delta t\left[\frac{d(Ma)}{dt}+\lambda\frac{d(Ma)}{dx}+\mu\frac{d(Ma)}{dy}+\nu\frac{d(Ma)}{dz}\right]\ldots(5)$$

and similar expressions for $M'\beta'$ and $M'\gamma'$.

Next, to determine the surface areas of the elements df_1', df_2', df_3', we project the surface-elements df and df_1', df_2', df_3' on their respective coordinate-planes. Take the projections of the elements df and df_1' on the yz-coordinate-plane; if we fix the corners of the former by the coordinates

$$(y, z),\ (y+dy, z),\ (y, z+dz),\ (y+dy, z+dz),$$

those of the latter will evidently be

$$(y+\mu\delta t, z+\nu\delta t),$$

$$\left[y+dy+\left(\mu+\frac{d\mu}{dy}dy\right)\delta t,\ z+\left(\nu+\frac{d\nu}{dy}dy\right)\delta t\right],$$

$$\left[y+\left(\mu+\frac{d\mu}{dz}dz\right)\delta t,\ z+dz+\left(\nu+\frac{d\nu}{dz}dz\right)\delta t\right],$$

$$\left[y+dy+\left(\mu+\frac{d\mu}{dy}dy+\frac{d\mu}{dz}dz\right)\delta t,\ z+dz+\left(\nu+\frac{d\nu}{dy}dy+\frac{d\nu}{dz}dz\right)\delta t\right]$$

or, if we refer them to a system of coordinates parallel to the given system but with its origin at the point

$$(y+\mu\delta t, z+\nu\delta t),$$

the following:

$$(0, 0),\ \left[dy\left(1+\frac{d\mu}{dy}\delta t\right),\ dy\frac{d\nu}{dy}\delta t\right],$$

$$\left[dz\frac{d\mu}{dz}\delta t,\ dz\left(1+\frac{d\nu}{dz}\delta t\right)\right],$$

$$\left[dy\left(1+\frac{d\mu}{dy}\delta t\right)+dz\frac{d\mu}{dz}\delta t,\ dz\left(1+\frac{d\nu}{dz}\delta t\right)+dy\frac{d\nu}{dy}\delta t\right].$$

As the area of the surface-element df_1', defined by these four points, differs from that of the parallelogram formed upon its two sides

$$(0,0)-\left[dy\left(1+\frac{d\mu}{dy}\delta t\right),\ dy\frac{d\nu}{dy}\delta t\right]$$

and

$$(0,0)-\left[dz\frac{d\mu}{dz}\delta t,\ dz\left(1+\frac{d\nu}{dz}\delta t\right)\right]$$

by quantities of higher infinitesimal order than those that determine the area of the latter, we can replace the given surface-element df_1' by this parallelogram. Since the area of the given parallelogram is now given by the following determinate, we can thus write

$$df_1'=\begin{vmatrix} dy\left(1+\frac{d\mu}{dy}\delta t\right), & dy\frac{d\nu}{dy}\delta t \\ dz\frac{d\mu}{dz}\delta t, & dz\left(1+\frac{d\nu}{dz}\right)\delta t \end{vmatrix}$$

$$=dy\,dz\left[1+\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\delta t+\left(\frac{d\mu}{dy}\frac{d\nu}{dz}-\frac{d\mu}{dz}\frac{d\nu}{dy}\right)\delta t^2\right]$$

or, if we reject quantities of the second infinitesimal order,

$$df_1'=dy\,dz\left[1+\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\delta t\right]=\left[1+\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\delta t\right]df.$$

Similarly, we find the following expressions for the areas of the surface-elements df_2' and df_3':

$$df_2'=-\frac{d\lambda}{dy}\delta t\,df,\quad df_3'=\frac{d\lambda}{dz}\delta t\,df. \qquad \ldots(6)$$

Finally, we substitute the above values (5) and (6) for $M'\alpha'$, $M'\beta'$, $M'\gamma'$ and df_1', df_2', df_3' respectively in the given expression (4) for δF, and we find, after rejecting all quantities of higher infinitesimal order than the first,

$$\delta F=\left[\frac{d(M\alpha)}{dt}+\lambda\frac{d(M\alpha)}{dx}+\mu\frac{d(M\alpha)}{dy}+\nu\frac{d(M\alpha)}{dz}\right.$$
$$\left.+M\alpha\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)-M\beta\frac{d\lambda}{dy}-M\gamma\frac{d\lambda}{dz}\right]df\,\delta t.$$

Lastly, replace δF in formulae (3) by this expression, and we get

$$\frac{d(M\alpha)}{dt}+\lambda\frac{d(M\alpha)}{dx}+\mu\frac{d(M\alpha)}{dy}+\nu\frac{d(M\alpha)}{dz}$$
$$+M\alpha\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)-M\beta\frac{d\lambda}{dy}-M\gamma\frac{d\lambda}{dz}=\mathfrak{A}\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)$$

or

$$\left.\begin{aligned}
&\frac{d(M\alpha)}{dt}+\lambda\left(\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right)\\
&\quad+\frac{d}{dy}M(\mu\alpha-\lambda\beta)+\frac{d}{dz}M(\nu\alpha-\lambda\gamma)=\mathfrak{A}\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)\\
&\text{and similarly}\\
&\frac{d(M\beta)}{dt}+\mu\left(\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right)\\
&\quad+\frac{d}{dz}M(\nu\beta-\mu\gamma)+\frac{d}{dx}M(\lambda\beta-\mu\alpha)=\mathfrak{A}\left(\frac{dP}{dz}-\frac{dR}{dx}\right),\\
&\frac{d(M\gamma)}{dt}+\nu\left(\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right)\\
&\quad+\frac{d}{dx}M(\lambda\gamma-\nu\alpha)+\frac{d}{dy}M(\mu\gamma-\nu\beta)=\mathfrak{A}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right).
\end{aligned}\right\}\quad(7)$$

These equations give the rates of change of α, β, γ at any point of space. If P, Q, R and α, β, γ are given at any period, we can determine by these equations the rates of change of α, β, γ for the period directly succeeding the given period, then from these new values for α, β, γ their rates of change in the next succeeding one, and in this manner, by continued repetition of this operation,

their rates of change at any later period—λ, μ, ν must, of course, be given here as functions of x, y, z and t; observe that we hereby avoid the actual solution of the equations themselves.

The derivation of equations for the determination of the rates of change of the quantities P, Q, R for moving bodies, to replace Maxwell's fundamental equations (9, II.), is exactly similar to the above. We conceive, namely, entire space to be traversed by the so-called lines of electric induction, whose direction and magnitude at every point are given by the value of the vector

$$D\sqrt{P^2+Q^2+R^2},$$

and determine the variation in the number of lines of electric induction that pass through the surface-elements df, dg, dh. If the given body were at rest, we should have as above

$$\delta F = D\delta P df = D\frac{dP}{dt}\delta t df$$

or, by formulae (9, II.),

$$\delta F = \left[\mathfrak{B}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)-4\pi L(P+X)\right]df\,\delta t \quad \ldots\ldots\ldots(8)$$

and similar expressions for δG and δH. We next make the analogous assumption to that on p. 420 that the lines of electric induction are also carried along through space with the moving body, that is, that these expressions for δF, δG, δH hold for bodies in motion.

Similarly, we then find a second set of values for δF, δG, δH, namely

$$\delta F = D'P'df_1' + D'Q'df_2' + D'R'df_3' - DPdf \quad \ldots\ldots(9)$$

and similar expressions for δG and δH, where, as above (cf. equations (5)),

$$D'P' = DP + \delta t\left[\frac{d(DP)}{dt}+\lambda\frac{d(DP)}{dx}+\mu\frac{d(DP)}{dy}+\nu\frac{d(DP)}{dz}\right], \quad (10)$$

with similar expressions for $D'Q'$ and $D'R'$; the surface-elements df_1', df_2', df_3' are those of the preceding development and are thus determined by the same equations (6).

Lastly, substitute the above values for δF, δG, δH in equations (8), and we get

$$\left.\begin{aligned}
&\frac{d(DP)}{dt}+\lambda\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]+\frac{d}{dy}D(\mu P-\lambda Q)\\
&\qquad+\frac{d}{dz}D(\nu P-\lambda R)=\mathfrak{B}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)-4\pi L(P+X)\\
&\text{and similarly}\\
&\frac{d(DQ)}{dt}+\mu\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]+\frac{d}{dz}D(\nu Q-\mu R)\\
&\qquad+\frac{d}{dx}D(\lambda Q-\mu P)=\mathfrak{B}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)-4\pi L(Q+Y),\\
&\frac{d(DR)}{dt}+\nu\left[\frac{d(DP)}{dx}+\frac{d(DQ)}{dy}+\frac{d(DR)}{dz}\right]+\frac{d}{dx}D(\lambda R-\nu P)\\
&\qquad+\frac{d}{dy}D(\mu R-\nu Q)=\mathfrak{B}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)-4\pi L(R+Z).
\end{aligned}\right\}(11)$$

These equations give the rates of change of P, Q, R at any point; as their general form is similar to that of equations (7), which give the rates of change of α, β, γ, the text to the latter will also apply here. The only real difference between these two systems of equations is, in fact, the appearance of the terms $4\pi L(P+X)=4\pi p$, $4\pi L(Q+Y)=4\pi q$ and $4\pi L(R+Z)=4\pi r$ in the latter, where p, q, r represent the components of the galvanic current.

To interpret the different terms of equations (7) and (11), we write them as follows:

$$\begin{aligned}
&4\pi\left[p+\frac{1}{4\pi}\frac{d(DP)}{dt}+\lambda\epsilon_r\right]\\
&\quad=\mathfrak{B}\left\{\frac{d}{dz}\left[\beta-\frac{D}{\mathfrak{B}}(\nu P-\lambda R)\right]-\frac{d}{dy}\left[\gamma-\frac{D}{\mathfrak{B}}(\lambda Q-\mu P)\right]\right\} \quad (12)
\end{aligned}$$

and

$$4\pi\left[\frac{1}{4\pi}\frac{d(Ma)}{dt}+\lambda\eta_r\right]$$
$$=\mathfrak{V}\left\{\frac{d}{dy}\left[R-\frac{M}{\mathfrak{V}}(\mu a-\lambda\beta)\right]-\frac{d}{dz}\left[Q-\frac{M}{\mathfrak{V}}(\lambda\gamma-\nu a)\right]\right\}, \quad (13)$$

with similar equations in Q, R and β, γ respectively, where ϵ_r and η_r denote the densities of the real electricity and magnetism respectively. It follows from this form of our equations that the magnetic and electric forces are diminished in magnitude by the quantities

$$a_1=\frac{D}{\mathfrak{V}}(\mu R-\nu Q),\ \ \beta_1=\frac{D}{\mathfrak{V}}(\nu P-\lambda R),\ \ \gamma_1=\frac{D}{\mathfrak{V}}(\lambda Q-\mu P)$$

and

$$P_1=\frac{M}{\mathfrak{V}}(\nu\beta-\mu\gamma),\ \ Q_1=\frac{M}{\mathfrak{V}}(\lambda\gamma-\nu a),\ \ R_1=\frac{M}{\mathfrak{V}}(\mu a-\lambda\beta)$$

respectively. P_1, Q_1, R_1 are the electric forces due to the motion of the given body through the magnetic field; we could thus designate them as the electromotive forces induced by motion. It is evident that their vector (P_1, Q_1, R_1) acts at right angles both to the direction of motion (λ, μ, ν) of the given body and to that of the lines of magnetic induction. The magnetic forces a_1, β_1, γ_1 are to be analogously interpreted.

The second terms of equations (12),

$$\frac{1}{4\pi}\frac{d(DP)}{dt},\ \ \frac{1}{4\pi}\frac{d(DQ)}{dy},\ \ \frac{1}{4\pi}\frac{d(DR)}{dz},$$

are identical to those that represent the components of the supplementary electric current of § 35 or to those of our electric polarization-current, provided we assume that the electrodynamic action of the latter is $\frac{D}{D-\mathfrak{e}}$ times that of the former (cf. p. 335). We have seen moreover

in § 35 that the electric polarization-currents were produced by the presence of variable electric currents in the given dielectric; the current in question is however due to the motion of the given body.

The third terms of equations (12), $\lambda\epsilon_r$, $\mu\epsilon_r$, $\nu\epsilon_r$, the density of the charge times the components of the velocity of translation, evidently correspond to an element of electric current, whose component current-densities are given by these expressions; these currents, known as convective currents, should therefore act like galvanic currents on the magnetic needle; this has in fact been confirmed by the experiments of Rowland* and Röntgen† with rapidly rotating ebonite discs. The corresponding terms of equations (13) are to be analogously interpreted; we observe that, strictly speaking, there is nothing in magnetism that corresponds to the galvanic current; the motion of magnetic poles or solenoids is, however, often referred to as the magnetic current.

Let us next examine the behaviour of the real electricity and magnetism in any system of conductors or magnets in motion. Take any volume-element

$$d\tau = dx\,dy\,dz$$

of the given system; the quantity of real magnetism contained in it will be

$$m = \eta_r d\tau, \quad \text{where} \quad \eta_r = \frac{1}{4\pi}\left[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\right].$$

The quantity of magnetism contained in the given volume-element after the elapse of time δt will be $m' = \eta_r' d\tau'$, where $d\tau'$ denotes the volume of the given element and η_r' the density of its real magnetism after the elapse of this time. Hereby the given volume-element will have undergone certain deformations, since the velocity of translation of its several parts will in

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* Rowland, *Poggendorf's Annalen*, 158, p. 487, 1876.

† Röntgen, *Annalen der Physik und Chemie*, 40, pp. 93-108, 1890.

general be an arbitrary function of x, y, z. If we denote the coordinates of the eight corners of the given element by

$$(x, y, z),\ (x+dx, y, z),\ (x, y+dy, z),\ (x, y, z+dz), \text{ etc.},$$

those of the corners of the element $d\tau'$ will evidently be

$$\left.\begin{array}{c}(x+\lambda\delta t,\ \ y+\mu\delta t,\ \ z+\nu\delta t),\\ \left[x+dx+\left(\lambda+\frac{d\lambda}{dx}dx\right)\delta t,\ y+\left(\mu+\frac{d\mu}{dx}dx\right)\delta t,\right.\\ \left.z+\left(\nu+\frac{d\nu}{dx}dx\right)\delta t\right], \text{ etc.,}\end{array}\right\}\ldots\text{(14)}$$

and hence the lengths of its sides

$$dx'^2 = dx^2\left\{\left(1+\frac{d\lambda}{dx}\delta t\right)^2+\left[\left(\frac{d\mu}{dx}\right)^2+\left(\frac{d\nu}{dx}\right)^2\right]\delta t^2\right\}, \text{ etc.};$$

if we reject here all quantities of higher infinitesimal order than the first, we have

$$dx' = dx\sqrt{1+2\frac{d\lambda}{dx}\delta t} = dx\left(1+\frac{d\lambda}{dx}\delta t\right)$$

and similarly

$$dy' = dy\left(1+\frac{d\mu}{dy}\delta t\right) \text{ and } dz' = dz\left(1+\frac{d\nu}{dz}\delta t\right).$$

The volume of the given volume-element will therefore be approximately

$$d\tau' = dx'dy'dz' = d\tau\left[1+\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\delta t\right]. \ldots\text{(15)}$$

The exact value of the volume of the parallelopiped $d\tau'$ would be given by the determinate

$$d\tau' = \begin{vmatrix} x_1'-x_0', & y_1'-y_0', & z_1'-z_0' \\ x_2'-x_0', & y_2'-y_0', & z_2'-z_0' \\ x_3'-x_0', & y_3'-y_0', & z_3'-z_0' \end{vmatrix},$$

where the indices 0, 1, 2, 3 refer to its four corners (x', y', z'), $(x'+dx', y', z')$, $(x', y'+dy', z')$ and $(x', y', z'+dz')$ respectively, or by formulae (14)

$$d\tau' = \begin{vmatrix} \left(1+\frac{d\lambda}{dx}\delta t\right)dx, & \frac{d\mu}{dx}\delta t\,dx, & \frac{d\nu}{dx}\delta t\,dx \\ \frac{d\lambda}{dy}\delta t\,dy, & \left(1+\frac{d\mu}{dy}\delta t\right)dy, & \frac{d\nu}{dy}\delta t\,dy \\ \frac{d\lambda}{dz}\delta t\,dz, & \frac{d\mu}{dz}\delta t\,dz, & \left(1+\frac{d\nu}{dz}\delta t\right)dz \end{vmatrix};$$

if we retain quantities of the null and first infinitesimal orders only, the value of this determinate evidently reduces to that already found.

The density of the real magnetism in the volume-element $d\tau'$ is now a function of x, y, z and t; if we put

$$\eta_r = f(x, y, z, t),$$

we have then

$$\begin{aligned} \eta_r' &= f(t+\delta t, x+\lambda\delta t, y+\mu\delta t, z+\nu\delta t) \\ &= \eta_r + \delta t\left[\frac{d\eta_r}{dt}+\lambda\frac{d\eta_r}{dx}+\mu\frac{d\eta_r}{dy}+\nu\frac{d\eta_r}{dz}\right]. \quad \text{.......(16)} \end{aligned}$$

The values (15) and (16) give the following expression for the quantity of real magnetism in the given element $d\tau'$:

$$m' = \eta_r' d\tau' = \eta_r d\tau + \left[\frac{d\eta_r}{dt}+\frac{d(\lambda\eta_r)}{dx}+\frac{d(\mu\eta_r)}{dy}+\frac{d(\nu\eta_r)}{dz}\right]\delta t\,d\tau.$$

The variation (increment) in the quantity of real magnetism in any volume-element $d\tau$ during the time δt will thus be

$$\delta m = m' - m = \left[\frac{d\eta_r}{dt}+\frac{d(\lambda\eta_r)}{dx}+\frac{d(\mu\eta_r)}{dy}+\frac{d(\nu\eta_r)}{dz}\right]\delta t\,d\tau. \quad \text{....(17)}$$

The value of the expression in the larger brackets must now be determined from our equations of action (7): for this purpose we differentiate them, the first with regard

to x, the second to y and the third to z, and add; we find then

$$\frac{d}{dt}\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]$$
$$+\frac{d}{dx}\left\{\lambda\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]\right\}$$
$$+\frac{d}{dy}\left\{\mu\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]\right\}$$
$$+\frac{d}{dz}\left\{\nu\left[\frac{d(M\alpha)}{dx}+\frac{d(M\beta)}{dy}+\frac{d(M\gamma)}{dz}\right]\right\}=0$$

or

$$\frac{d\eta_r}{dt}+\frac{d(\lambda\eta_r)}{dx}+\frac{d(\mu\eta_r)}{dy}+\frac{d(\nu\eta_r)}{dz}=0. \ldots\ldots(18)$$

By performing a similar operation on equations (11) we find

$$\frac{d\epsilon_r}{dt}+\frac{d(\lambda\epsilon_r)}{dx}+\frac{d(\mu\epsilon_r)}{dy}+\frac{d(\nu\epsilon_r)}{dz}$$
$$=-\left[\frac{d}{dx}L(P+X)+\frac{d}{dy}L(Q+Y)+\frac{d}{dz}L(R+Z)\right]. \quad (19)$$

By equation (18) the above expression (17) for δm evidently vanishes. Hence it follows that bodies containing magnetism will neither lose nor acquire it during any motion through space, a verification of the statement made in § 25, p. 234, that namely Maxwell's equations of electricity and magnetism for moving bodies likewise fail to account for the creation or destruction of real magnetism.

The behaviour of the real electricity in bodies in motion can be analogously examined; instead of equation (17) we find the following for δe:

$$\delta e=e'-e=\left[\frac{d\epsilon_r}{dt}+\frac{d(\lambda\epsilon_r)}{dx}+\frac{d(\mu\epsilon_r)}{dy}+\frac{d(\nu\epsilon_r)}{dz}\right]\delta t\, d\tau,$$

which by equation (19) can be written

$$\delta e = -\left[\frac{d}{dx}L(P+X)+\frac{d}{dy}L(Q+Y)+\frac{d}{dz}L(R+Z)\right]\delta t\, d\tau \quad \ldots(20)$$

For insulators, $L=0$, this expression for δe evidently vanishes; the behaviour of electricity in moving insulators will therefore be exactly similar to that of real magnetism (cf. preceding page). It is evident, however, that this analogy will not hold for conductors, $L \lesseqgtr 0$, for we recognize that the given expression for δe is then that for the diminution in the quantity of electricity in the given volume-element during the time δt due to the presence of the galvanic currents. Hence it follows that the real electricity in any system will behave the same, whether that system be in motion or at rest.

In the derivation of the above equations we have taken the surface-elements in their initial positions parallel to the three coordinate-planes respectively; we obtain an expression of a somewhat more general character upon determining the variation in the number of lines of induction that pass through any surface-element *do*—we shall denote the direction cosines of its normal by l, m, n. The number of lines of magnetic induction that pass through the given element is then evidently

$$N = M\alpha\, df + M\beta\, dg + M\gamma\, dh,$$

where df, dg, dh denote its projections on the coordinate-planes yz, xz and xy respectively. Next, let the time δt elapse, and denote the given surface-element (area) by do' and its three projections by df', dg', dh'—the latter will however no longer lie in planes parallel to the coordinate-planes. The number of lines of induction that pass through do' will then be

$$\left.\begin{array}{l} N' = M'\alpha'\, df' + M'\beta'\, dg' + M'\gamma'\, dh', \\ \text{where} \\ M'\alpha' = M\alpha + \delta t\left[\dfrac{d(M\alpha)}{dt} + \lambda\dfrac{d(M\alpha)}{dx} + \mu\dfrac{d(M\alpha)}{dy} + \nu\dfrac{d(M\alpha)}{dz}\right] \end{array}\right\}, \quad (21)$$

with similar expressions for $M'\beta'$ and $M'\gamma'$.

We next project the surface-elements df', dg', dh' on the coordinate-planes, and we have

$$df' = df_1' + dg_1' + dh_1'$$
$$dg' = df_2' + dg_2' + dh_2'$$
$$dh' = df_3' + dg_3' + dh_3',$$

where df_1', df_2', df_3' denote the three projections on df', etc. The values of these nine surface-elements are now given by formulae (6) and the following, obtained from the former by cyclic permutation:

$$dg_2' = \left[1 + \left(\frac{d\nu}{dz} + \frac{d\lambda}{dx}\right)\delta t\right] dg, \quad dh_3' = \left[1 + \left(\frac{d\lambda}{dx} + \frac{d\mu}{dy}\right)\delta t\right] dh,$$

$$dg_3' = -\frac{d\mu}{dz}\delta t\, dg, \qquad dh_1' = -\frac{d\nu}{dx}\delta t\, dh,$$

$$dg_1' = -\frac{d\mu}{dx}\delta t\, dg, \qquad dh_2' = -\frac{d\nu}{dy}\delta t\, dh.$$

Substituting these values for df_1', df_2', ..., dh_3' in the above expression (21) for N' and rejecting all quantities of higher infinitesimal order than the first, we find

$$\begin{aligned} N' = M\alpha\, df + \delta t\Big\{\lambda\Big[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\Big] \\ + \frac{d}{dy}M(\mu\alpha - \lambda\beta) + \frac{d}{dz}M(\nu\alpha - \lambda\gamma)\Big\} df \\ + M\beta\, dg + \delta t\Big\{\mu\Big[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\Big] \\ + \frac{d}{dz}M(\nu\beta - \mu\gamma) + \frac{d}{dx}M(\lambda\beta - \mu\alpha)\Big\} dg \\ + M\gamma\, dh + \delta t\Big\{\nu\Big[\frac{d(M\alpha)}{dx} + \frac{d(M\beta)}{dy} + \frac{d(M\gamma)}{dz}\Big] \\ + \frac{d}{dx}M(\lambda\gamma - \nu\alpha) + \frac{d}{dy}M(\mu\gamma - \nu\beta)\Big\} dh, \end{aligned}$$

and hence by formulae (7) the following value for δN:

$$\delta N = \mathfrak{V}\left[\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)df+\left(\frac{dP}{dz}-\frac{dR}{dx}\right)dg+\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)dh\right]\delta t.$$

Finally, replace here df, dg, dh by their values $l\,do$, $m\,do$, $n\,do$, and we get

$$\frac{\delta N}{\delta t} = \mathfrak{V}\left[\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)l+\left(\frac{dP}{dz}-\frac{dR}{dx}\right)m+\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)n\right]do. \quad (22)$$

We can now make use of this expression for determining the variation in the number of lines of induction ($\delta\Omega$) that pass through any finite surface or the aperture formed by any closed curve (circuit); the configuration of the circuit may hereby undergo any deformation whatever. The variation in the number of lines that pass through any given circuit during the time δt is now evidently given by the following integral:

$$\delta\Omega = \delta t\,\mathfrak{V}\iint\left[\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)l+\left(\frac{dP}{dz}-\frac{dR}{dx}\right)m+\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)n\right]do.$$

By Stokes' theorem this expression can be written

$$\delta\Omega = \delta t\,\mathfrak{V}\int(P\,dx+Q\,dy+R\,dz),$$

which gives

$$\int(P\,dx+Q\,dy+R\,dz)=\frac{1}{\mathfrak{V}}\,\frac{\delta\Omega}{\delta t}. \quad \text{.............}(23)$$

We recognize that this integral is the expression for the total electromotive force acting in the given circuit. It follows therefore that the electromotive force acting in the given circuit will alone be determined by the variation in the number of lines of induction passing through it, and hence that the phenomena of induced action for bodies in motion can be deduced from the same equation (14, XV.) that has already been employed as fundamental equation in our treatment of induced action for bodies at rest; as the derivation of the former

phenomena is thus precisely similar to that of the latter, we refer the reader here to the previous chapters on induction. We only observe that, when any closed circuit is rotated in a magnetic field, the variation in Ω will be greatest, as the circuit is entering or leaving any plane parallel to the direction of the lines of induction, and smallest, as it passes through the plane at right angles to that direction.

SECTION XLIII. INTRODUCTION INTO THE THEORY OF ELECTRO- AND MAGNETO-STRICTION; MAXWELL'S EQUATIONS FOR AEOLOTROPIC BODIES.

We shall conclude this treatise with a brief introduction into the theory of so-called electro- and magneto-striction, in which we can only attempt to make our reader familiar with the general conceptions and principles that form the fundament of this important and extensive subject and to establish the equations, from which the phenomena of electro- and magneto-striction can be obtained; the discussion of the equations and the phenomena expressed by them, a subject even more extensive than that of Maxwell's fundamental equations, must necessarily be left to the interested student.

We can define electro- and magneto-striction as the theory of the deformation of given bodies or media, due to the action of electric and magnetic forces residing within them, and of all changes or phenomena accompanying or succeeding such deformations. Take, as example, a sphere charged with electricity; as the electricity in any one of its surface-elements is evidently repelled by that in its other elements, the sphere itself will tend to expand radially. The magnetizing of a bar of soft iron should likewise give rise to similar expansions in its two ends; the expansion of a soap-bubble upon being charged is perceptible to the naked eye. To detect small expansions or contractions a

large glass bulb terminating in a narrow graduated glass tube is filled with a fluid and immersed with the tube uppermost in a second fluid; the first fluid is then connected with the one coating of a Leyden jar and the second fluid with its other coating. These two fluids thus become oppositely polarized, and hence attract each other; this attraction gives rise to a tangential force along the surface of the sphere; the latter expands, and the fluid sinks within the tube; by reading the initial and final values of the niveau on the graduated tube we can then determine quantitatively the given expansion. This experiment also serves to demonstrate that the phenomena of electro- and magneto-striction are not confined to solid bodies only, but that they play a most important rôle among the fluids. A phenomenon of the latter class would be the double refraction of light upon entering a fluid under electric (magnetic) stress, and conversely, its presence would confirm the action of electric (magnetic) forces within it.

We should not conclude that the above deformations and phenomena constitute alone the theory of electro- and magneto-striction. They form indeed only one such class; take, as example, of another class of these phenomena an iron rod set in rapid vibrations by an alternating current of suitably chosen period.

It is apparent from the above that the explanation of the phenomena of electro- and magneto-striction must be sought in the assumption of an aeolotropic ether. We must, therefore, first develop a system of equations that hold for such media; the equations of the preceding chapters hold for only isotropic not aeolotropic bodies. The changes to be made in Maxwell's fundamental equations must evidently follow from those assumed for the energies. Maxwell takes the following integral-expression as the kinetic energy of aeolotropic media:

$$T=\int\frac{d\tau}{8\pi}(D_1P^2+D_2Q^2+D_3R^2)$$

(cf. p. 5), where D_1, D_2, D_3 denote the constants of the electric induction along the principal axes, taken as coordinate-axes, of the given medium or body. If we refer our body to any system of coordinates, we must evidently write T as follows:

$$T=\int\frac{d\tau}{8\pi}$$

$$\times(D_{11}P^2+D_{22}Q^2+D_{33}R^2+2D_{12}PQ+2D_{13}PR+2D_{23}QR)\,..(24)$$

The expression in the brackets is a homogeneous function of the second degree in P, Q, R; since the D's must always be taken positive, it evidently represents an ellipsoid. We denote this function by $\phi(P, Q, R)$; differentiating it with regard to P, Q, R respectively, we have

$$\left.\begin{aligned}\frac{1}{2}\frac{d\phi}{dP}&=D_{11}P+D_{12}Q+D_{13}R=f\\ \frac{1}{2}\frac{d\phi}{dQ}&=D_{12}P+D_{22}Q+D_{23}R=g\\ \frac{1}{2}\frac{d\phi}{dR}&=D_{13}P+D_{23}Q+D_{33}R=h,\end{aligned}\right\}\ldots\ldots\ldots\ldots(25)$$

where f, g, h are at present only notations. These formulae give

$$fP+gQ+hR=\phi(P, Q, R);$$

we can thus write the above expression for T as follows:

$$T=\int\frac{d\tau}{8\pi}(fP+gQ+hR).$$

The analogous form of the expression for T for isotropic media would evidently be

$$T=\int\frac{d\tau}{8\pi}(DP\,.\,P+DQ\,.\,Q+DR\,.\,R).$$

From a comparison of these two expressions we next assume that the quantities f, g, h represent the com-

ponents of the electric polarization in aeolotropic media; the lines of electric induction are then determined by the vector (f, g, h).

Analogously, we assume the following expression for the magnetic energy V of aeolotropic media:

$$V=\int\frac{d\tau}{8\pi}$$
$$\times(M_{11}\alpha^2+M_{22}\beta^2+M_{33}\gamma^2+2M_{12}\alpha\beta+2M_{13}\alpha\gamma+2M_{23}\beta\gamma)$$
$$=\int\frac{d\tau}{8\pi}\psi(\alpha,\beta,\gamma)=\int\frac{d\tau}{8\pi}(a\alpha+b\beta+c\gamma), \quad\text{.................(26)}$$

where
$$\left.\begin{aligned}a&=M_{11}\alpha+M_{12}\beta+M_{13}\gamma\\ b&=M_{12}\alpha+M_{22}\beta+M_{23}\gamma\\ c&=M_{13}\alpha+M_{23}\beta+M_{33}\gamma\end{aligned}\right\}.\quad\text{.............(27)}$$

It is evident that the expression for Joule's heat, namely

$$J=\int[L(P+X)^2+L(Q+Y)^2+L(R+Z)^2]d\tau,$$

must also be generalized in a similar manner; as the constant of conductivity L plays here an analogous rôle to that of the constants D and M above, we must evidently write

$$\begin{aligned}J=\int[&L_{11}(P+X)^2+L_{22}(Q+Y)^2+L_{33}(R+Z)^2\\ &+2L_{12}(P+X)(Q+Y)+2L_{13}(P+X)(R+Z)\\ &+2L_{23}(Q+Y)(R+Z)]d\tau \quad\text{......................(28)}\\ =\int&\chi(R+X, Q+Y, R+Z)d\tau\\ =\int&[p(P+X)+q(Q+Y)+r(R+Z)]d\tau,\end{aligned}$$

where
$$\left.\begin{aligned}p&=L_{11}(P+X)+L_{12}(Q+Y)+L_{13}(R+Z)\\ q&=L_{12}(P+X)+L_{22}(Q+Y)+L_{23}(R+Z)\\ r&=L_{13}(P+X)+L_{23}(Q+Y)+L_{33}(R+Z)\end{aligned}\right\}.\text{..(29)}$$

The equations of electric and magnetic action for aeolotropic bodies at rest are evidently those already developed, provided we only replace the former variables DP, DQ, DR, $M\alpha$, $M\beta$, $M\gamma$ and $L(P+X)$, $L(Q+Y)$, $L(R+Z)$ by the new ones f, g, h, a, b, c and p, q, r respectively; we have then

$$\left.\begin{aligned}\frac{1}{\mathfrak{V}}\frac{df}{dt}&=\frac{d\beta}{dz}-\frac{d\gamma}{dy}-\frac{4\pi}{\mathfrak{V}}p\\ \frac{1}{\mathfrak{V}}\frac{dg}{dt}&=\frac{d\gamma}{dx}-\frac{d\alpha}{dz}-\frac{4\pi}{\mathfrak{V}}q\\ \frac{1}{\mathfrak{V}}\frac{dh}{dt}&=\frac{d\alpha}{dy}-\frac{d\beta}{dx}-\frac{4\pi}{\mathfrak{V}}r\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(30)$$

and

$$\left.\begin{aligned}\frac{da}{dt}=\mathfrak{V}\left(\frac{dR}{dy}-\frac{dQ}{dz}\right),\quad \frac{db}{dt}=\mathfrak{V}\left(\frac{dP}{dz}-\frac{dR}{dx}\right)\\ \frac{dc}{dt}=\mathfrak{V}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)\end{aligned}\right\},\ldots\ldots(31)$$

where f, g, h and a, b, c, which are given by formulae (25) and (27) respectively, define the electric and magnetic state of the given medium (ether).

Before developing the equations of electric and magnetic action for aeolotropic bodies in motion, let us confirm the validity of the principle of the conservation of energy for the above equations (for aeolotropic bodies at rest). The Joule's heat generated in any system is now given by the integral

$$J=\int[p(P+X)+q(Q+Y)+r(R+Z)]\,d\tau.\ldots..(32)$$

If the external electromotive forces have ceased to act, the total energy of the system must evidently be constant, and we thus have

$$dT+dV+J_0dt=\text{const.},$$

where J_0 denotes the value of J for $X=Y=Z=0$. If, however, external electromotive forces reside in the

system, the principle of the conservation of energy must evidently be written in the form

$$dT+dV+J\,dt=dt\int(pX+qY+rZ)\,d\tau$$

or
$$\frac{dT}{dt}+\frac{dV}{dt}+J=\int(pX+qY+rZ)\,d\tau, \text{ ....(33)}$$

where the integral-expression represents the total energy of the given external electromotive forces.

To prove the validity of the equation (33) of the conservation of energy, we replace T, V and J by their respective values (24), (26) and (32), and we have

$$\int\frac{d\tau}{8\pi}\frac{d}{dt}[D_{11}P^2+D_{22}Q^2+D_{33}R^2+2D_{12}PQ+2D_{13}PR+2D_{23}QR]$$

$$+\int\frac{d\tau}{8\pi}\frac{d}{dt}[M_{11}\alpha^2+M_{22}\beta^2+M_{33}\gamma^2+2M_{12}\alpha\beta+2M_{13}\alpha\gamma+2M_{23}\beta\gamma]$$

$$+\int d\tau[p(P+X)+q(Q+Y)+r(R+Z)]=\int d\tau[pX+qY+rZ],$$

hence

$$\int\frac{d\tau}{4\pi}\left[P\frac{df}{dt}+Q\frac{dg}{dt}+R\frac{dh}{dt}\right]+\int\frac{d\tau}{4\pi}\left[\alpha\frac{da}{dt}+\beta\frac{db}{dt}+\gamma\frac{dc}{dt}\right]$$

$$+\int d\tau[pP+qQ+rR]=0; \text{ ....(34)}$$

by formulae (30) the last integral of this equation can be written

$$\int d\tau[pP+qQ+rR]=-\int\frac{d\tau}{4\pi}\left\{P\left[\frac{df}{dt}-\mathfrak{B}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)\right]\right.$$

$$\left.+Q\left[\frac{dg}{dt}-\mathfrak{B}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)\right]+R\left[\frac{dh}{dt}-\mathfrak{B}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)\right]\right\}$$

$$=-\int\frac{d\tau}{4\pi}\left[P\frac{df}{dt}+Q\frac{dg}{dt}+R\frac{dh}{dt}\right]+\mathfrak{B}\int\frac{d\tau}{4\pi}\left[\left(R\frac{d\alpha}{dy}-Q\frac{d\alpha}{dz}\right)\right.$$

$$\left.+\left(P\frac{d\beta}{dz}-R\frac{d\beta}{dx}\right)+\left(Q\frac{d\gamma}{dx}-P\frac{d\gamma}{dy}\right)\right],$$

or, since, by partial integration and formulae (31), we have

$$\mathfrak{B}\int\frac{d\tau}{4\pi}\left[\left(R\frac{d\alpha}{dy}-Q\frac{d\alpha}{dz}\right)+\left(P\frac{d\beta}{dz}-R\frac{d\beta}{dx}\right)+\left(Q\frac{d\gamma}{dx}-P\frac{d\gamma}{dy}\right)\right]$$

$$=\mathfrak{B}\int\frac{d\tau}{4\pi}\left[\alpha\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+\beta\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+\gamma\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)\right]$$

$$=-\int\frac{d\tau}{4\pi}\left(\alpha\frac{da}{dt}+\beta\frac{db}{dt}+\gamma\frac{dc}{dt}\right),$$

as follow:

$$\int d\tau(pP+qQ+rR)=-\int\frac{d\tau}{4\pi}\left(P\frac{df}{dt}+Q\frac{dg}{dt}+R\frac{dh}{dt}\right)$$

$$-\int\frac{d\tau}{4\pi}\left(\alpha\frac{da}{dt}+\beta\frac{db}{dt}+\gamma\frac{dc}{dt}\right).$$

Lastly, substituting this value for the given integral in the above equation (34) of the conservation of energy, we see that it is identically satisfied.

Finally, to obtain the equations of electric and magnetic action for aeolotropic bodies in motion, we make the same assumptions and have recourse to the same conceptions and principles that have already been introduced in the preceding article on the derivation of Maxwell's equations for isotropic bodies in motion from those for isotropic bodies at rest.

If we refer our equations to a system of coordinates, for which $\lambda=\mu=\nu=0$, they assume a somewhat simpler form; we have then

$$F'+G'+H'=a'df_1'+b'df_2'+c'df_3',$$

where $a'=a+\dfrac{da}{dt}\delta t,\quad b'=b+\dfrac{db}{dt}\delta t,\quad c'=c+\dfrac{dc}{dt}\delta t;$

df_1', df_2' ... are given here by the same expressions (6)—$\lambda=\mu=\nu=0$—as above.

Retaining quantities of only the null and the first infinitesimal orders of magnitude, we next find the following expression:

$$F'+G'+H'=\left[a+\frac{da}{dt}\,\delta t\right]\left[1+\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)\delta t\right]df$$
$$-b\,\frac{d\lambda}{dy}\,\delta t\,df-c\,\frac{d\lambda}{dz}\,\delta t\,df,$$

hence $$\delta F=\delta t\left[\frac{da}{dt}+a\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)-b\frac{d\lambda}{dy}-c\frac{d\lambda}{dz}\right]df$$

and similar expressions for δG and δH.

Finally, these expressions for the component-variations in the number of lines of induction that pass through the given surface-elements give the following equations for the magnetic action in aeolotropic bodies in motion:

$$\frac{da}{dt}+a\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)-b\frac{d\lambda}{dy}-c\frac{d\lambda}{dz}=\mathfrak{B}\left(\frac{dR}{dy}-\frac{dQ}{dz}\right)$$

and, similarly,

$$\frac{db}{dt}+b\left(\frac{d\nu}{dz}+\frac{d\lambda}{dx}\right)-c\,\frac{d\mu}{dz}-a\frac{d\mu}{dx}=\mathfrak{B}\left(\frac{dP}{dz}-\frac{dR}{dx}\right) \qquad (35)$$

$$\frac{dc}{dt}+c\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}\right)-a\frac{d\nu}{dx}-b\,\frac{d\nu}{dy}=\mathfrak{B}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right).$$

Analogously, we find the following equations for their electric action:

$$\left.\begin{aligned}
&\frac{df}{dt}+f\left(\frac{d\mu}{dy}+\frac{d\nu}{dz}\right)-g\,\frac{d\lambda}{dy}-h\,\frac{d\lambda}{dz}=\mathfrak{B}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)-4\pi p\\
&\frac{dg}{dt}+g\left(\frac{d\nu}{dz}+\frac{d\lambda}{dx}\right)-h\frac{d\mu}{dz}-f\frac{d\mu}{dx}=\mathfrak{B}\left(\frac{d\gamma}{dx}-\frac{da}{dz}\right)-4\pi q\\
&\frac{dh}{dt}+h\left(\frac{d\lambda}{dx}+\frac{d\mu}{dy}\right)-f\frac{d\nu}{dx}-g\,\frac{d\nu}{dy}=\mathfrak{B}\left(\frac{da}{dy}-\frac{d\beta}{dx}\right)-4\pi r.
\end{aligned}\right\}(36)$$

The treatment of these equations of action (35) and (36) and of the expressions for the electric and magnetic energies (24) and (26)—the variations of the latter represent the phenomena of electro- and magneto-striction —is recommended to the student as continuation of the present theoretical treatise.

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